Stochastic Calculus, Courant Institute, Fall 2012

http://www.math.nyu.edu/faculty/goodman/teaching/StochCalc2012/index.html

Always check the class message board on the blackboard site from home.nyu.edu before doing any work on the assignment.

Assignment 4, due October 8

Corrections: (Oct. 2: Due date is October 8, parts (b) and (c) of question 3 were in the wrong order. The order is now correct.
Oct. 3: signs fixed in (1d) ($m_t \to -\infty$) and (1e) ($F = -Cu$).)

1. (*Brownian motion with reflection*) A *reflecting Brownian motion*, with a *reflecting barrier* at $x = a$, is a stochastic process that never crosses a and does not stick to a. For $X_t \neq a$, X_t acts like a Brownian motion. Suppose $X_0 = 0$ and $a > 0$. A reflecting Brownian motion has a probability density, $X_t \sim u(x,t)$, that satisfies the heat equation if $x < a$, and has

$$\int_{-\infty}^{a} u(x,t)\,dx = 1\ . \tag{1}$$

   (a) The conservation formula (1) implies a boundary condition that u satisfies at $x = a$. What is this condition? Hint: What must the probability flux be at $x = a$? This boundary condition is called a *reflecting* boundary condition. For `wikipedia` lovers, it is also called a *Neumann* boundary condition.

   (b) Show that if $v(x)$ is symmetric about the point a, which is the condition $v(a-x) = v(a+x)$ for all x, and if v is a smooth function of x, then v satisfies the boundary condition from part a.

   (c) Use the method of images from this week's material to write a formula for the $u(x,t)$ that satisfies the correct initial condition for $X_0 = 0$ and boundary condition at $a > 0$. It is closely related to the formula from class.

   (d) Write a formula for $m_t = \mathrm{E}[\,X_t]$ for reflecting Brownian motion. The *cumulative normal* distribution is $N(z) = \mathrm{P}(Z \leq z)$, when $Z \sim \mathcal{N}(0,1)$. Derive a formula for m_t in terms of this and other explicit functions. Verify that m_t is exponentially small for small t. Verify that $m_t \to -\infty$ as $t \to \infty$ and scales as $t^{1/2}$.

   (e) It is argued (possibly later in this course, or the book *Stochastic Integrals* by Henry McKean) that a reflecting Brownian motion is kept inside the allowed region $\{x \leq a\}$ by a rightward force at the reflecting boundary. This force is different from zero only when $X_t = a$. The force is just strong enough to prevent $X_t > a$. This picture suggests that the total force is proportional to the total time spent at the boundary. Since only the boundary force has a preferred direction, if $X_0 = 0$, it may be that

$$\mathrm{E}[\,X_t] = \mathrm{E}\left[\int_0^t F_s ds\right] \ ,$$

both sides being negative. Since the force only acts when $X_t = a$, it may be plausible that $\mathrm{E}[\,F_s] = -C\,u(a,s)$. Verify that this picture is true, at least as far as the formula

$$m_t = -C\int_0^t u(x,t)\,dt \ .$$

Find $C > 0$.

2. (*Kolmogorov reflection principle*) Let X_n be a discrete time *symmetric* random walk on the integers, positive and negative. The random walk is symmetric if $\mathrm{P}(x \to x+1) = \mathrm{P}(x \to x-1)$. Suppose the walk starts with $X_0 = 0$. Let $H_a(t) = \mathrm{P}(X_n = a$ for some $n \leq t)$ be the hitting probability for this discrete process. Show that if $a > 0$, then

$$\mathrm{P}(H_a(t)) = \mathrm{P}(X_t = a) + 2\mathrm{P}(X_t > a) \ . \qquad (2)$$

Hint: The discrete time version of the argument from class is rigorous.

3. (*Brownian bridge construction of Brownian motion*) Suppose X_t is a standard Brownian motion. Suppose $0 \leq t_1 < t_2 < t_3$.

    (a) Write the two dimensional PDF of (X_{t_2}, X_{t_3}) conditional on X_{t_1}. Call it $u(x_2, x_3, s_2, s_3|x_1)$, where x_j refers to the value of X_{t_j} and $s_1 t_2 - t_1$ and $s_2 = t_3 - t_2$. These are the time increments between t_1 and t_2, and t_2 and t_3, respectively.

    (b) Conditional on $X_{t_1} = x_1$ and $X_{t_3} = x_3$, find the distribution of X_{t_2}. This is $\mathcal{N}(\mu, \sigma^2)$ for some μ and σ^2 that depend on x_1, x_3, s_1, and s_2. Hint: The conditional density of X_{t_2} is the exponential of a quadratic. Identify the mean and variance by completing the square in the exponent.

    (c) Show that the formula of part (b) is the same as the conditional density of X_{t_2} given any number of additional values for times $t_k < t_1$ and/or $t_k > t_3$. For example, conditioning on $X_{t_4} = x_4$ with $t_4 > t_3$ does not change the answer to (b) in the sense that x_4 and t_4 do not appear in the answer.

    (d) Specialize to the case $s_1 = s_2 = \Delta t$. Compare the variance of X_{t_2} with both X_{t_1} and X_{t_3} specified to the variance with only X_{t_1} specified.

    (e) (*not an action item*) You can use these formulas to generate Brownian motion paths in a different way. First generate $X_1 \sim \mathcal{N}(0,1)$ and $X_0 = 0$. Then use the result of (d) to generate $X_{1/2} \sim \mathcal{N}(\cdot,\cdot)$

(results of (d)). Then use (d) again to generate $X_{1/4}$ using X_0 and $X_{1/2}$, and $X_{3/4}$ from $X_{1/2}$ and X_1. Continuing in this way you can make a Brownian motion path in as much detail as you want.

4. (*backward equation*) Let X_t be a standard Brownian motion starting from $X_0 = 0$. Let $\tau = \min\{t \text{ so that } |X_t| = 1\}$. Find the expected hitting time $\mathrm{E}[\tau]$. Hint:

   (a) Suppose $V(x,t)$ is a running time reward function and the total reward starting from x at time t is

   $$\int_t^\tau V(X_s, s)\, ds \;.$$

   There the process starts with $X_t = s$, and τ is the first hitting time after t, and $|x| \leq 1$. Define the value function for this to be

   $$f(x,t) = \mathrm{E}_{x,t}\left[\int_t^\tau V(X_s, s)\, ds \right] \;.$$

   Figure out the PDE that f satisfies.

   (b) The case $V(x,t) = 1$ gives the expected hitting time.

   (c) There is a subtlety here that we need to show $\mathrm{E}[\tau] < \infty$. The assignment for a future week will show that there is a $x > 0$ so that $\mathrm{P}_{x,0}(\tau > t) \leq e^{-ct}$.

5. (*Computing*) **New this week**: *Download the file* `coding.pdf`. *It contains guidelines for coding. Ultimately they will save you time in the computing assignments.* The material for this week contains the PDF

   $$M_t = \max_{0 \leq s \leq t} X_s$$

   and a formula for

   $$S_{t,a}(x)dx = \mathrm{P}(x \leq X_t \leq x + dx \mid X_s < a \text{ for } 0 \leq s \leq t)$$

   You made histograms of these distributions last week. This week, put the exact formulas on the graphs to see whether they agree. Play with parameters to see how good a fit you can get in a reasonable amount of computer time.