

Assignment 4.

Given October 1, due October 14. Last revised, October 1.

Objective: Gaussian random variables.

1. Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Find a formula for $E[e^X]$. Hint: write the integral and complete the square in the exponent. Use the answer without repeating the calculation to get $E[e^{aX}]$ for any constant a .
2. In finance, people often use $N(x)$ for the CDF (cumulative distribution function) for the standard normal. That is, if $Z \sim \mathcal{N}(0, 1)$ then $N(x) = P(Z \leq x)$. Suppose $S = e^X$ for $X \sim \mathcal{N}(\mu, \sigma^2)$. Find a formula for $E[\max(S, K)]$ in terms of the N function. (Hint: as in problem 1.) This calculation is part of the Black–Scholes theory of the value of a vanilla European style call option. K is the known strike price and S is the unknown stock price.
3. Suppose $X = (X_1, X_2, X_3)$ is a 3 dimensional Gaussian random variable with mean zero and covariance

$$E[XX^*] = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Set $Y = X_1 + X_2 - X_3$ and $Z = 2X_1 - X_2$.

- a. Write a formula for the probability density of Y .
- b. Write a formula for the joint probability density for (Y, Z) .
- c. Find a linear combination $W = aY + bZ$ that is independent of X_1 .
4. Take $X_0 = 0$ and define $X_{k+1} = X_k + Z_k$, for $k = 0, \dots, n-1$. Here the Z_k are iid. standard normals, so that

$$X_k = \sum_{j=0}^{k-1} Z_j. \quad (1)$$

Let $X \in R_n$ be the vector $X = (X_1, \dots, X_n)$.

- a. Write the joint probability density for X and show that X is a multivariate normal. Identify the $n \times n$ tridiagonal matrix H that arises.
- b. Use the formula (1) to calculate the variance of X_k and the covariance $E[X_j, X_k]$.
- c. Use the answers to part b to write a formula for the elements of H^{-1} .
- d. Verify by matrix multiplication that your answer to part c is correct.