Practice Final Exam questions.

Given December 11, last revised December 12

Focus: Review

This should help you study for the final exam. There are more questions than could probably be on the actual final in order to review more material.

1. In each part, state whether the statement is true or false and give a short explination with a reason it is true (maybe something short of a full mathematical proof) or a counterexample.

a. Suppose $dX_t = (X_t^3 - 1)dt + X_t^2 dW_t$ with $X_0 = 2$. It is possible to compute an accurate approximation to $E[X_T^2]$ with $T = 3$ without simulating a random process.

b. Suppose $X(\omega)$ and $Y(\omega)$ are functions of the random variable ω defined for $\omega \in \Omega$, a probability space. Let $\mathcal{F}_X$ and $\mathcal{F}_Y$ be the $\sigma-$algebras generated respectively by X and Y. Let $\mathcal{F}_{X,Y}$ be the $\sigma-$algebra generated by both X and Y. Then $\mathcal{F}_{X,Y} = \mathcal{F}_X \cup \mathcal{F}_Y$.

c. If $(X_t^{(1)}, X_t^{(2)})$ is a two component Markov process, then $X_t^{(1)}$ separately is a one component Markov process.

d. If $f(x,t)$ satisfies the PDE

$$\partial_t f + (2 + \sin(x))\partial_x^2 f = 0 \ ,$$

and f is bounded, then

$$f(x,t) \leq \max_{x'} f(x', T) \ ,$$

whenever $T > t$.

e. If X_t and Y_t are independent Brownian motions, then $X_T^2 Y_T - \int_0^T Y_t dt$ is a martingale.

f. If $\sigma_t \in \mathcal{F}_t$ and $dX_t = \sigma_t dW_t$ then X_t is a Markov process.

g. If there is a function $b(x)$ so that $\sigma_t = b(X_t)$ and $dX_t = \sigma_t dW_t$, then X_t is a Markov process.

h. If X is a two dimensional diffusion process that satisfies $dX_t = a(X_t)dt + \sigma(X_t)dW_t$, where W_t is a pair of independent Brownian motions, and $dY_t = a(Y_t)dt + \sigma(Y_t)QdW_t$ where Q is a 2×2 orthogonal matrix, then the probability measure on the path space $C([0,T] \to R^2)$ defined by X_t and Y_t are the same even though $X_t \neq Y_t$ almost surely.

2. Suppose X_t is standard Brownian motion and that $\mathcal{G}$ is the $\sigma-$ algebra generated by X_1 and X_2 (X evaluated at times $t = 1$ and $t = 2$). Let $Y = \int_0^3 X_t dt$. Write a formula for the conditional PDF $u(y \mid \mathcal{G}_0)$. This will be a function of the three variables y, x_1, and x_2. Hint: You know that the conditional probability densities multivariate normals is normal, so figure out something about the joint PDF of X_1, X_2, and Y. You need not write the joint PDF in complete detail.

3. Suppose $(X_1, X_2, X_3) \in R^3$ is a three dimensional multivariate normal with $E[X_1] = 1$, $E[X_2] = -1$, and $E[X_3] = 3$ and covariance matrix

$$C = \begin{pmatrix} 5 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} .$$

Write a formula for the joint PDF, $u(x_1, x_2, x_3)$. Hint: $C^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{pmatrix}$.

4. Suppose $dX_t = \mu X_t dt + \sigma X_t dW_t$ where μ and σ are constants. Suppose $V(x) = \max(x - K, 0)$ and that we want to evaluate $E[V(X_T)M_T]$ where $M_T = \exp(-r \int_0^T X_t dt)$. Write a PDE we could use.

5. We wish to solve the PDE $\partial_t f + \frac{x^2+y^2}{2}\partial_x^2 f + (x - y^2)\partial_y f + ryf = 0$, where r is some constant, and $f(x, y, T) = V(x, y)$ is given. Write an SDE and express $f(x, y, 0)$ as the expectation of some function of the path X_t, Y_t.

6. Suppose $dX_t = \sigma X_t dW_t$ with constant σ and $X_0 = 1$. Express

$$Y_T = \int_0^T X_t dX_t$$

as a function of X_T and the random variable

$$U_T = \int_0^T X_t^2 dt .$$

Verify that $E[Y_T] = 0$ (why is Y_T a martingale?) by calculating the expected values of the two terms $f(X_t)$ and U_T.

7. The state of a discrete time finite state space Markov chain is X_t at time $t = 0, 1, 2, \ldots$. The state space is $\mathcal{S} = \{0, 1, \ldots, n\}$. For $x \neq 0, n$ we the transition probabilities are $\frac{1}{3}$ to increase by one $X \rightarrow X + 1$, decrease by one, or stay the same. When $X = 0$, the transition probabilities are $\frac{1}{2}$ to go to one $X_t = 0 \rightarrow Xt + 1 = 1$ or stay the same. For $X = n$ the transition $X_t = n \rightarrow X_{t+1} = n$ has probability $\frac{2}{3}$ and the transition $X_t = n \rightarrow X_{t+1} = n - 1$ has probability $\frac{1}{3}$.

a. For $n = 4$, write the 4×4 transition matrix.

b. For $n = 4$, use two matrix multiplications to calculate all the time 4 transition probabilities $a_{jk} = P(X_{t+4} = k \mid X_t = j)$. (This might be too much arithmetic for the actual exam.)

c. We construct a sequence of stopping times $\tau_1 = \min(t$ such that $X_t = 0$ or $n)$, and $\tau_{k+1} = \min(t > \tau_k$ such that $X_t = 0$ or $n)$. The sequence $Y_k = X_{\tau_k}$ consists of the 0 and n values of X_t with all other values taken out. Show that the sequence Y_k is a two state space Markov chain. This fact is a special case of what is called the "strong Markov property".

d. Calculate the 2×2 transition matrix for the Y_k chain. To make it easier, replace the X transition probabilities at 0 by $P(X_{t+1} = 0 \mid X_t = 0) = \frac{2}{3}$ and $P(X_{t+1} = 1 \mid X_t = 0) = \frac{1}{3}$. This makes the X transition matrix (and therefore the Y transition matrix) symmetric. Do not assume $n = 4$.

8. Let V_t satisfy the familiar SDE $dV_t = -\gamma V_t dt + \alpha dW_t$. Let be the first hitting time for 0, $\tau = \min(t$ such that $V_t = 0$. Let $s = \min(\tau, 5)$. Let $\mathcal{F}_s$ be the $\sigma-$algebra generated by all V_t for $t \leq s$. Calculate $G = E[V_5^2 \mid \mathcal{F}_s]$ by showing that it is given as a simple function of one random variable.

9. Let X_t be Brownian motion starting at a point $X_0 > 0$ and let A_T be the event $X_t > 0$ for all $t \in (0, T)$. Let $u_0(x,t)$ be the probability density for paths in A_t to land at x at time t: $u_0(x,t)dx = P(X_T \in A_t \text{ and } x < X_t < x + dx)$.

a. Write the partial differential equation we can solve to determine $u(x,t)$ including initial conditions and boundary conditions.

b. Write a formula for the solution of this PDE as a sum (difference) of two gaussian functions.

c. Suppose instead that $dX_t = vdt + dW_t$ (Brownian motion with constant drift velocity, v). Write the PDE (with initial and boundary conditions) that determines $u_v(x,t)dx = P(X_T \in A_t \text{ and } x < X_t < x + dx)$.

d. Use Girsanov's theorem to express $u_v(x,t)$ in terms of $u_0(x,t)$.

10. Suppose $W_t^{(1)}$ and $W_t^{(2)}$ are Brownian motions with correlation coefficient ρ. For any two random variables X and Y, $\rho(X,Y) = \text{cov}(X,Y)/\sqrt{\text{var}(X)\text{var}(Y)}$. Here, we suppose that $\rho(W_t^{(1)}, W_t^{(2)}$ is a constant independent of t. We have the pair of SDE's

$$\begin{aligned} dX_t &= r_t X_t dt + \sigma X_t dW_t^{(1)} \\ dr_t &= \mu(\overline{r} - r_t)dt + \alpha\sqrt{r_t}dW_t^{(2)} . \end{aligned}$$

We want to compute

$$F = E[V(X_T)e^{-\int_0^T r_t dt}] .$$

Define a quantity $f(x,r,t)$ as a conditional expectation value and give a backward equation satisfied by f so that $f(x,r,0)$ is F.