Assignment 7.

Given October 31, due November 7 Last revised November 5.

Objective: The Ito integral

1. We want to calculate from the definition

$$Y_T = \int_0^T X_t^2 dX_t \ ,$$

where X_t is standard Brownian motion. Were X_t a differentiable function of t, the answer would be $Y_T = \frac{1}{3}X_T^3$. There will be an Ito correction to this. To find the correction, use the notations $X_k = X_{t_k}$, $t_k = k\Delta t$, and $\Delta X_k = X_{k+1} - X_k$, then calculate

$$X_{k+1}^3 - X_k^3 = 3X_k^2\Delta X_k + 3X_k\Delta X_k^2 + \Delta X_k^3 \ .$$

Turning this around gives

$$X_k^2\Delta X_k = \frac{1}{3}(X_{k+1}^3 - X_k^3) - X_k\Delta X_k^2 - \frac{1}{3}\Delta X_k^3 \ .$$

Thus

$$Y_T^{(n)} = \sum_{k=0}^{n-1} X_k^2\Delta X_k = \frac{1}{3}X_T^3 - A_n - B_n - \frac{1}{3}C_n \ ,$$

where

$$\begin{aligned} A_n &= \sum_{k=0}^{n-1} X_k\Delta t \ , \\ B_n &= \sum_{k=0}^{n-1} X_k(\Delta X_k^2 - \Delta t) \ , \\ C_n &= \sum_{k=0}^{n-1} \Delta X_k^3 \ . \end{aligned}$$

Find the limit as $\Delta t \to 0$ of A_n as an integral involving X_t. Show that $B_n \to 0$ and $C_n \to 0$ as $n \to \infty$ with $n = 2^L$ by calculating $E[B_n^2]$ and $E[C_n^2]$. Note that if $\sum_n E[B_n^2] < \infty$ then $\sum_n B_n^2 < \infty$ almost surely (why?), which implies that $B_n^2 \to 0$ as $n \to \infty$ almost surely.

2. Use the Ito isometry to calculate $E[Y_T^2]$, where

$$Y_T = \int_{t=0}^T \left(\int_{s=0}^t (t-s)X_s dX_s \right) dX_t \ .$$

It may help to find a forumla for $E[Z^4]$ and $E[Z^6]$ where $Z \sim \mathcal{N}(0,\sigma^2)$.

3. We want simple approximations to the integral

$$Y_T = \int_0^T V(X_t)dX_t$$

when T is small. For any random variable, W_T, we say that $W_T = O(t^p)$ if $E[|W_T|] = O(\Delta t^p)$. Note that this is not the same as saying that there is a constant C so that $|W_T| \leq Ct^p$. For example, if X_T is standard Brownian motion, we already used the fact that $X_T = O(\sqrt{T})$.

a. Use the Cauchy Schwartz inequality to show that if $W_T^2 = O(\Delta t^{2p})$ then $W_T = O(\Delta t^p)$.

b. Use the Ito isometry formula to show that if $F_t \in \mathcal{F}_t$, $F_t^2 = O(t^p)$, and $W_T = \int_0^T F_t dX_t$, then $W_t = O(t^{(p+1)/2})$.

c. Suppose $V(x)$ is a smooth function of x and take enough Taylor series terms of V to show that

$$Y_T = V(X_0)\Delta X_T + \frac{1}{2}V'(X_0)(\Delta X_T^2 - T) + O(T^{3/2}) \; .$$

We suppose the Brownian motion path X_t starts at X_0, which is not necessarily zero. We use the notation $\Delta X_T = X_T - X_0$. Hint: integrate the Taylor series for $V(X_t)$ term by term, and use parts a-c to get a bound for what we get by integrating the remainder term.

4. Find a backwards equation for

$$f(x,t) = E_{x,t}\left[\exp\left(\int_{s=t}^T V(X_s)dX_s\right)\right] \; .$$

Hint: First derive the formula using the tower property for conditional expectation,

$$f(x,t) = E_{x,t}\left[\exp\left(\int_{s=t}^{t+\Delta t} V(X_s)dX_s\right) \cdot f(x+\Delta X, t+\Delta t)\right] \; ,$$

then use Taylor series, including the answer to question 3.