Assignment 5.

Given October 19 due October 24 Last revised, October 23.

Objective: Simple martingales and PDE's for Brownian motion

1. Suppose $h(x)$ has $h'(x) > 0$ for all x so that there is a unique x for each y so that $y = h(x)$. Consider the process $Y_t = h(X_t)$, where X_t is standard Brownian motion. Suppose the function $h(x)$ is smooth. The answers to the questions below depend at least on second derivatives of h.

a. With the notation $\Delta Y_t = Y_{t+\Delta t} - Y_t$, for a positive Δt, calculate $a(y)$ and $b(y)$ so that $E[\Delta Y_t \mid \mathcal{F}_t] = a(Y_t)\Delta t + O(\Delta t^2)$ and $E[\Delta Y_t^2 \mid \mathcal{F}_t] = b(Y_t)\Delta t + O(\Delta t^2)$.

b. With the notation $f(Y_t, t) = E[V(Y_T) \mid \mathcal{F}_t]$, find the backward equation satisfied by f.

c. Writing $u(y,t)$ for the probability density of Y_t, *use the duality argument* to find the forward equation satisfied by u.

d. Write the forward and backward equations for the special case $Y_t = e^{cX_t}$. Note (for those who know) the similarity of the backward equation to the Black Scholes partial differential equation. We will get it exactly right soon.

2. Use a calculation similar to the one we used in class to show that $Y_T = X_T^4 - 6\int_0^T X_t^2 dt$ is a martingale. Here X_t is Brownian motion.

3. Show that $Y_t = \cos(kX_t)e^{k^2t/2}$ is a martingale. *Hint 1*: It suffices to show that $E[Y_{t+\Delta t} \mid \mathcal{F}_t] = Y_t + O(\Delta t^2)$, which is a calculation in this case. Now, to show $E[Y_{t'} \mid \mathcal{F}_t] = Y_t$ for $t' > t$, divide the interval (t, t') into n intervals of size $\Delta t = (t' - t)/n$ and use the tower property for the expanding family $\mathcal{F}_{t_k} \subset \mathcal{F}_{t_{k+1}}$, with $t_k = t + k\Delta t$. This will show that $E[Y_{t'} \mid \mathcal{F}_t] = Y_t + O(n\Delta t^2)$. The error term goes to zero as $n \to \infty$. *Hint 2*: Use the fact that a certain function satisfies the backward equation.

4. *Fourier solution of the heat equation*

a. Show that

$$\int_{-\pi/2}^{\pi/2} \cos(kx)\cos(jx)dx = \frac{\pi}{2}\delta_{jk}$$

if j and k are positive integers. This is the "orthogonality" relation between cosines.

b. Suppose $f(x) = \sum_{k=1}^{\infty} a_k \cos(kx)$ (a "Fourier series"). Show that

$$a_j = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos(jx)f(x)dx \ ,$$

assuming the sum converges rapidly enough. The numbers a_j are the "Fourier coefficients" of f.

c. Find a formula for $\int_{-\pi/2}^{\pi/2} \cos(jx)u(x,t)dx$, where $u(x,t)$ is the density for Brownian motion X_t with $X_0 = 0$ and $\tau > t$ with τ being the hitting time of $\pm\frac{\pi}{2}$. This can be done by using the result of question 3 and the observation that $\int \cos(jx)u(x,t)dx = E[\cos(jY_t)] = E[\cos(jX_t)]$ (Y_t being the stopped Brwonian motion).

d. This gives a representation $u(x,t) = \sum_{k=1}^{\infty} a_k(t)\cos(kx)$. Show by direct calculation that each of the terms satisfies the u forward equation and boundary conditions.

e. Conclude that for large t, $u(x,t) \approx e^{-t/2}\cos(x)$ and $P(\tau > t) \approx e^{-t/2}$.