# Assignment 2.

Given September 12, due September 19.
Last revised, September 15.

**Objective:** Conditioning and Markov chains.

**1.** Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two algebras of sets and that $\mathcal{G}$ adds information to $\mathcal{F}$ in the sense that any $\mathcal{F}$ measureable event is also $\mathcal{G}$ measurable. Since $\mathcal{F}$ and $\mathcal{G}$ are collections of events, this may be written $\mathcal{F} \subset \mathcal{G}$. Suppose that the probability space $\Omega$ is finite and that $X(\omega)$ is a variable defined on $\Omega$ (that is, a function of the random variable $\omega$). The conditional expectations (in the modern sense) of $X$ with respect to $\mathcal{F}$ and $\mathcal{G}$ are $Y = E[X \mid \mathcal{F}]$ and $Z = E[X \mid \mathcal{G}]$. In each case below, state whether the statement is true or false and explain your answer with a proof or a counterexample.

**a.** $Z \in \mathcal{F}$.

**b.** $Y \in \mathcal{G}$.

**c.** $Z = E[Y \mid \mathcal{G}]$.

**d.** $Y = E[Z \mid \mathcal{F}]$.

**2.** For any event $A \subseteq \Omega$ we can define the "indicator function" (also called the "characteristic function", particulatly by people who learned probability late in life), $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ if $\omega \notin A$. People who call this the "characteristic function" (usually people who learned probability late in life) write $\chi_A(\omega)$ for $\mathbf{1}_A(\omega)$.

**a.** Show that $E[\mathbf{1}_A] = P(A)$.

**b.** For any event, $B$, show that the classical $E[\mathbf{1}_A \mid B]$ is the same as the Bayes' rule conditional probability of $A$.

The "modern" definition of conditional probability, conditioning on an algebra of sets rather than a single set, is $P(A \mid \mathcal{F}) = E[\mathbf{1}_A \mid \mathcal{F}]$. The relationship between the classical and modern definition of conditional probability is more or less the same as the relation between classical and modern expected value.

**3.** Let $\mathcal{S}$ be a finite state space and $\Omega$ be the set of paths of length $T$ from $\mathcal{S}$. Let $P(X)$ be the probability of a path $X \in \Omega$. For any $t$ in the range $1 < t < T$, let $\mathcal{F}_t$ be the algebra of events in $\Omega$ generated by the values of $X_s$ for $1 \leq s \leq t$. Let $\mathcal{G}_t$ be the smaller algebra generated only by $X_t$. Finally, let $\mathcal{H}_t$ be the "complementary" algebra (based on complementary information) generated by the values $X_t$, ..., $X_T$. An event in $\mathcal{H}_t$ is a statement about the path from time $t$ on without saying anything about the beginning values $X_1$, ..., $X_{t-1}$. Show that the Markov property is equivalent to either of the following,

**a.**

$$E[\mathbf{1}_A \mid \mathcal{F}_t] = E[\mathbf{1}_A \mid \mathcal{G}_t] \quad \text{for any } A \in \mathcal{H}_{t+1}.$$

**b.**

$$E[F(X) \mid \mathcal{F}_t] = E[F(X) \mid \mathcal{G}_t] \quad \text{for any } F \in \mathcal{H}_{t+1}.$$

**Notes:**(i) Part a is a special case of part b (why?). (ii) Part b implies that $E[F(X) \mid \mathcal{F}_t]$ is a function of $X_t$ only. This us supposed to be intuitively clear as a consequence of the Markov property. (iii) Together with question 1, part b is a justification for the backward equation for expected values of final payouts.

**4.** Suppose we have a 3 state Markov chain with transition matrix

$$P = \begin{pmatrix} .6 & .2 & .2 \\ .3 & .5 & .2 \\ .1 & .2 & .7 \end{pmatrix}$$

and suppose that $X_1 = 1$.

**a.** Show that the probability distribution of the first $t$ steps conditioned on $\mathcal{G}_{t+1}$ is the same as that conditioned on $\mathcal{H}_{t+1}$. This is a kind of backwards Markov property: a forward Markov chain is a backward Markov chain also.

**b.** Calculate $P(X_2 = 2 \mid \mathcal{G}_3)$. This consists of 3 numbers.