**Stochastic Calculus - Problem set 2 - Fall 2002**

**Exercise 1 - a**

This is false.

Since $Z = E[X|\mathcal{G}]$, $Z$ is $\mathcal{G}$-measurable, but there is no reason for $Z$ to be $\mathcal{F}$-measurable. Let us construct a counter-example. We choose $\Omega$ to be $\Omega = \{a, b, c\}$, $P$ is defined as $P(\{a\}) = P(\{b\}) = P(\{c\}) = \frac{1}{3}$ and $X : \Omega \to \mathbb{R}$ is such that $X(a) = 1$, $X(b) = 2$ and $X(c) = 3$. Of all the $\sigma$-algebra one can define on $\Omega$, we choose two very simple ones, $\mathcal{F} = \{\emptyset, \Omega\}$ and $\mathcal{G} = \{\emptyset, \Omega, \{a\}, \{b, c\}\}$.Any $\mathcal{F}$-measurable function has to be constant, so if $Y = E[X|\mathcal{F}]$, we necessarily have $Y = E(X) = \frac{1}{3}(1 + 2 + 3) = 2$. On the other hand, denoting by $Z = E[X|\mathcal{G}]$, we know that for each $\omega \in \Omega$, we have:

$$Z(\omega) = \begin{cases} E[X|\{a\}] & \text{if } \omega = \{a\} \\ E[X|\{b, c\}] & \text{if } \omega \in \{b, c\} \end{cases}$$

Since each event has the same probability, it is easy to see that:

$$Z(\omega) = \begin{cases} 1 & \text{if } \omega = \{a\} \\ \frac{2+3}{2} = \frac{5}{2} & \text{if } \omega \in \{b, c\} \end{cases}$$

Therefore $Z$ is not constant, and can not be $\mathcal{F}$-measurable.

**Exercise 1 - b**

This is true.

$Y = E[X|\mathcal{F}]$, thus $Y$ is $\mathcal{F}$-measurable. But $\mathcal{F} \subset \mathcal{G}$, therefore $Y$ is $\mathcal{G}$-measurable as well.

**Exercise 1 - c**

This is false.

We just proved that $Y$ is $\mathcal{G}$-measurable, and therefore $E[Y|\mathcal{G}] = Y$ almost surely. But the counter-example of part a) clearly shows the statement $Z = Y$ is false.

**Exercise 1 - d**

This is true.

First it is obvious that both random variables $Y$ and $E[Z|\mathcal{F}]$ are $\mathcal{F}$-measurable. Now let us pick any element $A$ of the $\sigma$-algebra $\mathcal{F}$. On one hand we have:

$$E[Y\mathbf{1}_A] = E[E[X|\mathcal{F}]\mathbf{1}_A]$$

and since $A \in \mathcal{F}$, we know $\mathbf{1}_A$ is $\mathcal{F}$-measurable, and the above expression is equal to:

$$E[Y\mathbf{1}_A] = E[E[X\mathbf{1}_A|\mathcal{F}]] = E[X\mathbf{1}_A]$$

On the other hand, because $\mathbf{1}_A$ is $\mathcal{F}$-measurable, we have:

$$E[E[Z|\mathcal{F}]\mathbf{1}_A] = E[E[Z\mathbf{1}_A|\mathcal{F}]] = E[Z\mathbf{1}_A]$$

But $Z = E[X|\mathcal{G}]$, and it follows from $\mathcal{F} \subset \mathcal{G}$ that $\mathbf{1}_A$ is $\mathcal{G}$-measurable as well, and therefore:

$$E[E[X|\mathcal{G}]\mathbf{1}_A] = E[E[X\mathbf{1}_A|\mathcal{G}]] = E[X\mathbf{1}_A]$$

Thus we have $E[Y\mathbf{1}_A] = E[E[Z|\mathcal{F}]\mathbf{1}_A]$ and both $Y$ and $E[Z|\mathcal{F}]$ are $\mathcal{F}$-measurable. By uniqueness of the conditional expecation, we have $Y = E[Z|\mathcal{F}]$ almost surely.

**Exercise 2 - a**
By definition $E\left[\mathbf{1}_A\right] = \sum_{\omega\in\Omega} \mathbf{1}_A(\omega) dP(\omega) = \sum_{\omega\in A} dP(\omega) = P(A)$.

**Exercise 2 - b**
It is the same proof:

$$E\left[\mathbf{1}_A|\mathcal{B}\right] = \sum_{\omega\in\Omega} \mathbf{1}_A(\omega) dP(\omega|\mathcal{B}) = \frac{1}{P(B)} \sum_{\omega\in\Omega} \mathbf{1}_A(\omega)\mathbf{1}_B dP(\omega) = \frac{1}{P(B)} \sum_{\omega\in A\cap B} dP(\omega) = \frac{P(A\cap B)}{P(B)}.$$

**Exercise 3 - a**
Property b) is one of the definition of the Markov property. First let us prove that a) and b) are equivalent. By choosing a specific $F(X) = \mathbf{1}_A(X)$, we see that b) implies a). We know the state space is finite, so we can write it as $S = \{a_1, \ldots, a_N\}$. Since $X$ takes its values in $S$, we have for any function $F$:

$$F(X) = \sum_{i=1,\ldots,N} F(a_i)\mathbf{1}_{\{a_i\}}(X)$$

Now we use the result a), and the linearity of the conditional expectation to get:

$$E\left[F(X)|\mathcal{F}_t\right] = \sum_{i=1,\ldots,N} F(a_i) E\left[\mathbf{1}_{\{a_i\}}(X)|\mathcal{F}_t\right] = \sum_{i=1,\ldots,N} F(a_i) E\left[\mathbf{1}_{\{a_i\}}(X)|\mathcal{G}_t\right] = E\left[F(X)|\mathcal{F}_t\right]$$

**Exercise 4 - a**
The state space is finite, so we can write it as $S = \{1, 2, 3\}$. We want to prove that for any $\{a_i\} \in S$ we have:

$$P(X_1 = a_1, \ldots, X_t = a_t | X_{t+1} = a_{t+1}, \ldots, X_T = a_T) = P(X_1 = a_1, \ldots, X_t = a_t | X_{t+1} = a_{t+1})$$

We have to transform the expression in order to use the regular Markov property. From now on, to make the notations lighter, we will write $P(X_1, \ldots, X_t)$ instead of $P(X_1 = a_1, \ldots, X_t = a_t)$. The left handside of the above expression is:

$$P(X_1, \ldots, X_t | X_{t+1}, \ldots, X_T) = \frac{P(X_1, \ldots, X_T)}{P(X_{t+1}, \ldots, X_T)} = \frac{P(X_{t+2}, \ldots, X_T | X_{t+1} \ldots, X_1) P(X_{t+1}, \ldots, X_1)}{P(X_{t+1}, \ldots, X_T)}$$

Then we use the regular Markov property on the upper-left part of this equation to get:

$$\frac{P(X_{t+2}, \ldots, X_T | X_{t+1}) P(X_{t+1}, \ldots, X_1)}{P(X_{t+1}, \ldots, X_T)} = \frac{P(X_{t+2}, \ldots, X_T | X_{t+1}) P(X_{t+1}, \ldots, X_1)}{P(X_{t+2}, \ldots, X_T | X_{t+1}) P(X_{t+1})}$$

Crossing out the identical terms, we obtain:

$$\frac{P(X_{t+1}, \ldots, X_1)}{P(X_{t+1})} = P(X_1, \ldots, X_t | X_{t+1})$$

**Exercise 4 - b** This is a simple calculation. I will give the details for the first one, the other ones are similar. We have:

$$P(X_2 = 2 | X_3 = 1) = \frac{P(X_2 = 2, X_3 = 1)}{P(X_3 = 1)} = \frac{P(X_3 = 1 | X_2 = 2) P(X_2 = 2)}{P(X_3 = 1)}$$

But the Markov chain is stationnary, therefore $P(X_3 = 1 | X_2 = 2) = P_{2,1}$ and since $X_1 = 1$, it follows:

$$P(X_2 = 2 | X_3 = 1) = \frac{P_{2,1} P_{1,2}}{P(X_3 = 1)}$$

There are 2 ways to calculate $P(X_3 = 1)$, either by conditionning upon $X_2$ and then we have:

$$P(X_3 = 1) = \sum_{i=1,\dots,3} P(X_3 = 1 | X_2 = i) P(X_2 = i) = \sum_{i=1,\dots,3} P_{i,1} P_{1,i}$$

or by directly using the fact that the transition probability matrix for $X_3$ is $P^2$. Either way, we find:

$$P(X_2 = 2 | X_3 = 1) = \frac{P_{2,1} P_{1,2}}{\sum_{i=1,\dots,3} P_{i,1} P_{1,i}} = \frac{0.3 * 0.2}{0.6 * 0.6 + 0.3 * 0.2 + 0.1 * 0.2} = 0.136364$$

For the 2 other possible states for $X_3$ we get:

$$P(X_2 = 2 | X_3 = 2) = \frac{0.5 * 0.2}{0.6 * 0.2 + 0.2 * 0.5 + 0.2 * 0.2} = 0.384615$$

and

$$P(X_2 = 2 | X_3 = 3) = \frac{0.2 * 0.2}{0.6 * 0.2 + 0.2 * 0.2 + 0.2 * 0.7} = 0.133333$$