FFT and applications*

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1 Background

Fourier series and integrals are the main technical tools in many aspects of science and engineering. The fast Fourier transform, or FFT, allows these to be the basis of efficient computational algorithms. This chapter has a quick review of Fourier series. It discusses the discrete version, the discrete Fourier transform, or DFT. It explains the fast algotihm for computing the DFT, the FFT. Finally, it samples some of the many applications.

The complex exponential,

$$e^{it} = \cos(t) + i\sin(t) \quad ,$$

simplifies the discussions below quite a bit. In particular, it saves us from writing sin and cos sums separately, and from using trigonometric identities. Some simple facts are $e^{i\pi} = -1$, $e^{2i\pi} = 1$, and $e^{i\pi/2} = i$. The formula $e^{a+b} = e^a \cdot e^b$ applies here. For example $e^{i(t+\pi)} = e^{it} \cdot e^{i\pi} = -e^{it}$. The rules for differentiating and integrating apply:

$$\frac{d}{dt}e^{int} = ine^{int}$$

$$\int_a^b e^{int}dt = \frac{1}{in}\left(e^{ib} - e^{ia}\right) \quad \text{if } n \neq 0.$$

If $z = x + iy$ is a complex number, then the complex conjugate is $\bar{z} = x - iy$, and $|z|^2 = x^2 + y^2 = \bar{z} \cdot z$. The complex conjugate of e^{it} is e^{-it}.

The rules for summing geometric series also apply to complex exponentials. The basic rule is that, for an integer n,

$$e^{na} = (e^a)^n \quad .$$

Consider the geometric series

$$S = z^k + z^{k+1} + \cdots + z^n \quad .$$

Multiplying this by z gives

$$sS = z^{k+1} + z^{k+2} + \cdots + z^{n+1} \quad .$$

Subtracting these expressions gives

$$S - zS = z^k - z^{n+1} \quad .$$

Solving for S gives the eventual formula for the geometric sum

$$S = z^k + z^{k+1} + \cdots + z^n = \frac{z^k - z^{n+1}}{1 - z} \quad \text{if } z \neq 1.$$

We will apply this formula in the case $z = e^{it}$ to find that

$$1 + e^{it} + e^{2it} + \cdots + e^{i(n-1)t} = 0 \quad \text{if } nt \text{ is an integer multiple of } 2\pi. \tag{1}$$

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2 Fourier series

Fourier series is a technique for analysing periodic functions. A function is periodic with period T if $f(t+T) = f(t)$ for all t. The graph of such a function repeats itself with repetition interval t. For example, the function $f(t) = \sin^2(t)$ is periodic with period π. This function is also periodic with period[1] 2π. If α is an integer, each of the functions $f(t) = e^{i\alpha t}$ is periodic with period 2π. Some functions are periodic by their nature. For example, if $f(t)$ is the temperature at the outer edge of a round disk at angle offset t from some point on the edge; then t and $t + 2\pi$ refer to the same place. Other functions happen to be periodic or nearly so. If $f(t)$ is the pressure as a function of time for a piano playing a single note, then $f(t)$ is very nearly periodic, with periodic $T = (1/440)$ *sec* for the note **A** below middle **C**. This function is not exactly periodic because the tone is slowly decaying, and because there may be static in the recording.

For a while, we suppose that $f(t)$ is periodic with period $T = 2\pi$. Each of the functions e^{int} has this property. A *Fourier series* is a sum of such functions:

$$f(t) = \sum_{\alpha} \hat{f}_\alpha e^{i\alpha t} \quad . \tag{2}$$

Although most of the terms in (2) has periods smaller than 2π, the sum probably does not. The sum is over all integer values of α, both positive and negative. The numbers $\hat{f}_\alpha$ are the *Fourier coefficients* of the function f. The functions $f(t) = e^{i\omega t}$ are sometimes called "pure tones" with frequency ω. For example, taking[2] $\omega = 2\pi/(440sec)$ gives the pure tone of **A** below middle **C**. The sound formed by $e^{2i\omega t}$ is an octave higher than that of $e^{i\omega t}$. The sound formed by $e^{3i\omega t}$ is an octave and a fifth higher than that of $e^{i\omega t}$. Each of the "higher harmonics", $e^{-i\alpha\omega t}$, for integer α, is on some musical relation to the "fundamental", $e^{i\omega t}$. The expression (2) is said to express the "tone" $f(t)$ as a sum including a fundamental and a possibly infinite series of harmonics. For this reason, the study of Fourier series and related topics is often called harmonic analysis. Like music, its high points are entertaining and its mastery is time consuming.

The remarkable fact is that any periodic function can be written as a Fourier series of the form (2). This fact was discovered Euler and popularized by Fourier as a method for solving partial differential equations. We will justify this claim later on. For now, I want to take it for granted and work out some consequences. The first consequence is the formula for the Fourier coefficients, $\hat{f}_\alpha$. This formula relies on the fact that, for any integer, γ,

$$\int_0^{2\pi} e^{i\gamma t} dt = \begin{cases} 2\pi & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases} \tag{3}$$

Now multiply (2) by $e^{-i\beta t}$ and integrate each term, to get

$$\int_0^{2\pi} e^{-i\beta t} f(t) dt = \sum_{\alpha} \hat{f}_\alpha \int_0^{2\pi} e^{i(\alpha-\beta)t} dt \quad .$$

Because of the relation (3), only one of the terms on the right side is different from zero. This leads to the simple formula

$$\hat{f}_\beta = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\beta t} f(t) dt \quad . \tag{4}$$

The simplicity of this formula makes the Fourier series representation (2) a practical tool.

The second consequence of the representation (2) is very important in the theory of Fourier series and will help us understand the mathematical structure of the discrete Fourier transform, or DFT. For this, it is better not to assume that $f(t)$ is real. Then $|f|^2 = \bar{f} \cdot f$. Let us take the complex conjugate of both sides of (2) and use a different summation index. This gives

$$\bar{f}(t) = \sum_{\beta} e^{-i\beta t} \bar{\hat{f}}_\beta \quad .$$

[1] If we want *the* period of a periodic function, meaning the smallest T so that f has period T, we will say so explicitly.

[2] The units of ω are different from those of T. The seconds go in the denominator here but were in the numerator for T.

Now multiply this by (2), integrate over t and sum over both α and β. Again, (3) implies that only terms involving $\bar{\hat{f}}_\alpha \hat{f}_\alpha = \left|\hat{f}_\alpha\right|^2$ are different from zero. This leads to the formula:

$$\int_0^{2\pi} |f(t)|^2 \, dt = 2\pi \sum_\alpha \left|\hat{f}_\alpha\right|^2 \quad . \tag{5}$$

This formula says that the "energy" in a signal (the left side of (5)) is the same as the sum of the squares of its Fourier coefficients.

A particularly entertaining application of the formula (5) was, I believe, first noticed by Euler. Suppose we take $f(t)$ to be the periodic "sawtooth" function whose values are given by $f(t) = t$ for $0 \leq t < 2\pi$. Calculating both sides of (5) gives:

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6} \quad .$$

It is relatively easy to explain to a smart high school student who knows geometry and algebra what both sides of this formula mean. How would you explain the reason for believing it is true?

2.1 Differentiation and qualitative properties of Fourier series

One of the primary applications of Fourier series, and the one that Fourier himself was interested in, is to the solution of differential equations. For this, it is important to know the relation between the Fourier coefficients of $f'(t)$ and those of $f(t)$. Even if we are not interested in differential equations, this relation helps understand many qualitative features of Fourier series that are important for applications. The formula is gotten by integration by parts:

$$\hat{f'}_\alpha = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\alpha t} f'(t) dt = i\alpha \frac{1}{2\pi} \int_0^{2\pi} e^{-i\alpha t} f(t) dt = i\alpha \hat{f}_\alpha \quad . \tag{6}$$

This formula may be iterated to give, for example,

$$\hat{f''}_\alpha = i\alpha \hat{f'}_\alpha = (i\alpha)^2 \hat{f}_\alpha = -\alpha^2 \hat{f}^2 \quad .$$

These relations allows us to solve certain different equations using Fourier series.

This differentiation formula has the following consequence: *the Fourier coefficients decay to zero very rapidly if and only if the function is very smooth.* For example, suppose that f is differentiable. Then the Fourier coefficients, $\hat{f'}_\alpha$ are bounded, and the coefficients of f itself, $\hat{f}_\alpha = \frac{1}{i\alpha}\hat{f'}_\alpha$, decau to zero at least as fast as $1/|\alpha =|$. If f' is also differentiable, so that f is twice differentiable, then $\left|\hat{f}_\alpha\right|$ decays to zero at least as fast as $1/\alpha^2$, and so on. A function that has many derivatives will have Fourier coefficients that to to zero faster than high powers of $1/|\alpha|$.

Conversely, if the Fourier coefficients go to zero faster than a high power of $1/|\alpha|$ then f has many bounded derivatives. For example, if $\left|\hat{f}_\alpha\right| \leq C/\alpha^4$, then

$$|f''(t)| = \left|\sum_\alpha -\alpha^2 \hat{f}_\alpha\right| \leq \sum_{\alpha \neq 0} C/\alpha^2 < \infty \; .$$

so f'' is bounded.

3 Discrete Fourier Transform, DFT

The discrete Fourier transform may be thought of as a discrete approximation to the integrals (4). Suppose we take N points uniformly spaced in the interval $[0, 2\pi]$. The spacing will be $\Delta t = 2\pi/N$ and the points may be taken to be

$$t_k = k\Delta t = 2\pi k/N \quad \text{for } k = 0, 1, \ldots, N-1. \tag{7}$$

The rectangle rule approximation to (4) is

$$\tilde{f}_\beta = \frac{1}{2\pi}\Delta t \sum_{k=0}^{N-1} e^{-i\beta t_k} f(t_k) \quad .$$

Now suppose we do not know the function $f(t)$ but only the samples $f_k = f(t_k)$. Then we can rewrite this as

$$\tilde{f}_\beta = \frac{1}{N}\sum_{k=0}^{N-1} \exp(-2\pi i\beta k/N) f_k \quad . \tag{8}$$

The vector of N discrete Fourier coefficients $\tilde{f} = (\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{N-1})$ is the DFT of the N samples $f = (f_0, f_1, \ldots, f_{N-1})$. Since the formulae (8) are linear, we can write this in matrix notation as

$$\tilde{f} = Wf \quad , \tag{9}$$

where W is an $N \times N$ matrix whose (α, j) entry is the complex number $\frac{1}{N}\exp(-2\pi i\beta k/N)$.

The periodicity properties of the DFT are more symmetric than those of the continuous Fourier series. The "forward" transform starts with N samples $f = (f_0, \ldots, f_{N-1})$ and produces the numbers $\tilde{f}_\alpha$ by the formula (8). We often think of the samples, f_k, as representing a discretely defined but periodic function of k. That is, we think of f_k as defined for all k with the understanding that $f_{k\pm N} = f_k$ for all k. The DFT formula (8) produces $\tilde{f}_\alpha$ with the same property:

$$\tilde{f}_{\alpha+N} = \tilde{f}_\alpha \qquad \text{for any } \alpha \text{ and } N.$$

Because of this periodicity, $\tilde{f}$ is represented by N values. However, precisely which α values we choose may differ from situation to situation. In the formal algebraic discussion, which we are in now, the choice $\alpha = 0, 1, \ldots, N-1$ is convenient. Later, when we are studying approximation properties, it will be better to use $-N/2 < \alpha \le N/2$. How we treat the endpoints $-N/2$ and $N/2$ depends on whether N is even or odd. In any case the DFT produces an N dimensional vector from an N dimensional vector.

The Fourier series was a way to represent a function of t in terms of the Fourier coefficients, $\hat{f}$. The DFT comes with a similar representation. Finding the representation amounts to finding the inverse of W. This is where (5) comes in. If we temporarily postulate a discrete analogue, it would relate the sum of the $\left|\tilde{f}_\alpha\right|^2$ to the sum of $\left|f_k\right|^2$. Such a relation would imply that the matrix W is unitray, or very nearly so. A unitary matrix would have the property that

$$W^*W = WW^* = I \qquad \textit{(conjectural and not exactly true).}$$

Here W^* is the conjugate transpose of W. The (k, α) entry of W^* is $\bar{w}_{\alpha,k}$, the complex conjugate of the (α, k) entry of W. All of this is a motivation for seeking the inverse of W by computing W^*W.

The (j, k) element of W^*W is

$$\begin{aligned}(W^*W)_{jk} &= \sum_{\alpha=0}^{N-1} \frac{1}{N}\exp(2\pi i\alpha j/N)\frac{1}{N}\exp(-2\pi i\alpha k/N) \\ &= \frac{1}{N}\left[\frac{1}{N}\sum_{\alpha=0}^{N-1}\exp(2\pi i\alpha(j-k)/N)\right] \quad .\end{aligned}$$

The sum in square braces is of the form (1) with $t = 2\pi(j-k)/N$. Therefore, the sum will be zero unless t is an integer multiple of 2π. Since j and k are both integers in the range $[0, 1, \ldots, N-1]$, t can be a multiple of 2π only if $j = k$. In case $j = k$, the sum in square brackets is one. That is

$$(W^*W)_{jk} = \begin{cases} \frac{1}{N} & \text{if } j = k, \\ 0 & \text{if } j \ne k. \end{cases}$$

In matrix form, this is

$$W^*W = \frac{1}{N} I \quad ,$$

which means that

$$W^{-1} = NW^* \quad .$$

Writing out the relation $f = NW^*\tilde{f}$ in component form gives the desired relation

$$f_k = \sum_\alpha \exp(2\pi i\alpha k/N)\tilde{f}_\alpha = \sum_\alpha e^{i\alpha t_k}\tilde{f}_\alpha \quad . \tag{10}$$

the second form makes clear the relation between the DFT representation of the discrete f and the Fourier series representation of the continuous f. Also, note the periodicity of f. If the $\tilde{f}_\alpha$ are any N numbers, then the numbers defined by (10) have the property that $f_{k+N} = f_k$ for any integer k.

For smooth functions, the continuous Fourier coefficients, $\hat{f}_\alpha$, and the DFT coefficients, $\tilde{f}_\alpha$, agree to high order. This may be seen using the following "aliasing" formula:

$$\tilde{f}_\alpha = \sum_{p=-\infty}^{\infty} \hat{f}_{\alpha+pN} \quad . \tag{11}$$

To prove (11), we substitute the Fourier series representation,

$$f_k = f(t_k) = \sum_\beta e^{i\beta t_k}\hat{f}_\beta + \sum_\beta \exp(2\pi i\beta k/N)\hat{f}_\beta \quad ,$$

into the DFT formula (8). This gives

$$\tilde{f}_\alpha = \frac{1}{N}\sum_{k=0}^{N-1} \exp^{-2\pi i\alpha k/N} \sum_\beta \exp(2\pi i\beta k/N)\hat{f}_\beta = \sum_\beta \left[\frac{1}{N}\sum_{k=0}^{N-1} \exp(2\pi i(\beta-\alpha)k/N)\right]\hat{f}_\beta \quad .$$

The sum inside square brackets is again a geometric series of the type (1). The sum will be zero if $(\beta-\alpha)/N$ is not an integer, and one of $(\beta-\alpha)/N = p$ is an integer. Summing up the non zero terms gives (11).

Now suppose α is small and that the Fourier coefficients of f decay rapidly to zero.[3] Then the largest term on the right side of (11) is the one with $p = 0$. The sum of the terms with $p \neq 0$ represents the error, the difference between $\tilde{f}_\alpha$ and $\hat{f}_\alpha$. Suppose that $\left|\hat{f}_\alpha\right| \leq C/\left|\alpha\right|^r$ with $r \geq 2$. Then the error term has the bound

$$\left|\hat{f}_\alpha - \tilde{f}_\alpha\right| = \left|\sum_{p\neq 0} \hat{f}_{\alpha+pN}\right| \leq 2\sum_{p=1}^{\infty} \frac{C}{N^r p^r} \leq \frac{C'}{N^r} \quad .$$

This shows that if f has r derivatives, than $\left|\hat{f}_\alpha - \tilde{f}_\alpha\right| = O(\Delta t^r)$ (remember that $\Delta t = 2\pi/N$). This is the technical basis of the claim that the DFT coefficients and the Fourier coefficients are very close to each other for large N and smooth f.

Let me emphasize this last point by giving it a name. The fact that $\tilde{f} - \hat{f} = O(N^{-r})$ for sufficiently smooth f for any r is an example of the phenomenon called "spectral accuracy". Approximations based on harmonics are often called "spectral methods", probably because harmonics are eigenfunctions (eigenvectors) and the set of eigenvalues is often called the spectrum. Spectral methods are more accurate, in the sense of formal order of accuracy, than any finite difference approximation. This makes spectral methods the method of choice for many applications. One of the main drawbacks of spectral methods is that the accuracy depends very much on the smoothness of f. If f is discontinuous at a single point, then spectral approximations can revert to first order in the entire computational domain. The process of spreading of errors away from the source (the discontinuity in this case) is called "pollution". There are spectral smoothing methods that can mitigate pollution to some degree.

[3] This can only be because f is smooth.

4 The FFT algorithm

People look for ways to incorporate the DFT in computational algorithms because the DFT is faster to compute than the formulae (8) or (10) suggest. Naive programming of (8) would lead to N multiplies for each α, so that computing all N components of $\tilde{f}$ would require $O(N^2)$ work. The fast Fourier transform, or FFT, computes the same thing in $O(N\log(N))$ work. The basic FFT algorithm discussed below was invented and popularized in the $1950'^s$ by Cooley and Tukey. Since then there have been many variants and extensions proposed. Some mathematicians attribute the algorithm to Gauss, who did indeed use the basic factorization idea in his study of quadratic reciprocity.[4] I do not think Gauss know the $N\log(N)$ aspect, which, for us, is most important.

The FFT is a "divide and conquer" algorithm. That is, a problem of size N is reduced to two problems of size $N/2$ and some postprocessing. Let $P(N)$ be the work to compute a size N FFT. The reduction leads to

$$P(N) = 2P(N/2) + C\cdot N \quad . \tag{12}$$

Iterating this relation gives

$$\begin{aligned} P(N) &= 2\left(2P(N/4) + C\cdot N/2\right) + C\cdot N \\ &= 4P(N/4) + C\cdot N + C\cdot N \\ &\;\cdot \\ &\;\cdot \\ &\;\cdot \\ P(N) &= NP(1) + \overbrace{C\cdot N + \cdots + C\cdot N}^{\log_2(N)\text{ terms}} \\ P(N) &= O(N\log(N)) \end{aligned}$$

What underlies (12) is some interesting manipulations with exponentials. Suppose that $N = 2M$ is an even number. From the vector f we form two half size vectors g and h with the even and odd components of f respectively:

$$g_k = f_{2k} \quad k = 0,\ldots,M-1 \;,$$

$$h_k = f_{2k+1} \quad k = 0,\ldots,M-1 \;.$$

The two size $M = N/2$ DFT's of g and h are given by

$$\tilde{g}_\alpha = \frac{1}{M}\sum_{k=0}^{M-1}\exp\left(\frac{-2\pi i\alpha k}{M}\right) g_k \;,$$

$$\tilde{h}_\alpha = \frac{1}{M}\sum_{k=0}^{M-1}\exp\left(\frac{-2\pi i\alpha k}{M}\right) h_k \;.$$

Once we have $\tilde{g}$ and $\tilde{h}$, we can construct $\tilde{f}$ in $O(N)$ operations. This is because (after some manipulation)

$$\tilde{f}_\alpha = \frac{1}{2}\left[\frac{1}{M}\sum_{k=0}^{M-1}\exp\left(\frac{-2\pi i\alpha 2k}{2M}\right) g_k + \frac{1}{M}\sum_{k=0}^{M-1}\exp\left(\frac{-2\pi i\alpha(2k+1)}{2M}\right) h_k\right]$$

so

$$\tilde{f}_\alpha = \frac{1}{2}\left(\tilde{g}_\alpha + \exp\left(\frac{-2\pi i\alpha}{N}\right)\tilde{h}_\alpha\right) \quad . \tag{13}$$

If N is a power of 2, we may compute the DFT of f using two size $N/2$ DFT's, for g and h. These may, in turn, be computed in terms of four DFT's of size $N/4$, and so on. More sophisticated versions of the FFT

[4] Suppose p and q are prime numbers. If there is an integer, x, so that $x^2 \equiv q \bmod p$, isthere an integer, y, so that $y^2 \equiv p \bmod q$? The answer, which depends in a simple way on p and q, is Gauss' famous law of quadratic reciprocity.

may be applied if N is not a power of 2. The simplest extension is for $N = M \cdot q$ where q is a prime number. In this case, we can reduce the DFT of f to q DFT's of size M. What we did above is just the case $q = 2$. This works best when q is small. Now try to think of an $O(N \log(N))$ algorithm that computes the DFT when N is a prime number.

5 Applications

I will give just two applications. This is a pity because there are so many more. Spectral interpolation and differentiation are applications of the fact that $\hat{f}_\alpha \approx \tilde{f}_\alpha$. The FFT applications in signal processing and statistics are based on the fact that discrete convolutions may be computed fast using the FFT. Signal processors use convolutions to filter signals, because convolutions are the only time invariant linear operations.

Spectral interpolation is very simple. We have N samples of our function, $f(t)$ at the evenly spaced points t_k. Our approximation is

$$F(t) = \sum_{\alpha=0}^{N-1} e^{i\alpha t} \tilde{f}_\alpha \quad .$$

Because the DFT coefficients are spectrally accurate approximations to the actual Fourier coefficients, and because the Fourier coefficients not included are very small, F is a spectrally accurate approximation to f, provided that f is smooth. We can also use this idea to estimate derivatives of f, either at the sample points or elsewhere.

to be continued....