Scientific Computing Assignment 6, sample exam questions

1. A linear recurrence relation is a formula of the form

$$a_0 u_n + \cdots + a_p u_{n+p} = 0 \; . \tag{1}$$

The recurrence relation is *order* p and involves $p+1$ terms if $a_0 \neq 0$ and $a_p \neq 0$.

(a) Choose $N > p$. Let V be the vector space of sequences $\vec{u} = (u_0, u_1, \ldots, u_N)$ that satisfy (1) for $0 \leq n \leq N-p$. Show that if the recurrence relation is order p then V is a vector space of dimension p. Hint: Once the initial sequence $u_0, \ldots, u_{p-1}$ is chosen, the rest of the u_n are determined, and the initial sequence may be chosen arbitrarily.

(b) The characteristic polynomial is

$$f(z) = a_0 + a_1 z + \cdots + a_p z^p \; . \tag{2}$$

Show that if $f(z_k) = 0$ then $u_n = z_k^n$ gives $\vec{u} \in V$.

(c) Let $\vec{u}_k$ be the sequence above. Show that if $z_1, \ldots, z_p$ are distinct roots ($f(z_k) = 0$, $z_j \neq z_k$ if $j \neq k$) then the $\vec{u}_k \in V$ form a basis for V. Show that this implies that any solution of the recurrence relation (1) may be written $u_n = \sum_{k=1}^{p} a_k z_k^n$. Hint: You need to show they are linearly independent. You can do this by showing that the vanderMonde matrix is non singular.

(d) We say z_k is a root of multiplicity m if $f(z_k) = 0$, $f'(z_k) = 0$, $\ldots$, $f^{(m-1)}(z_k) = 0$, but $f^{(m)}(z_k) \neq 0$. Show that the sequences $\vec{u}_{k,j}$ with

$$u_n = n^j z_k^n \; , \tag{3}$$

for $0 \leq n \leq N-p$, have $\vec{u}_{k,j} \in V$. Hint: Define $\vec{u}(z)$ to have elements $u_n(z) = z^n$. Then $\vec{u}(z) \in V \iff f(z) = 0$. Furthermore, $\frac{d}{dz}\vec{u}(z) \in V \iff f'(z) = 0$, etc. You can express (3) as a linear combination of $\left(\frac{d}{dz}\right)^j \vec{u}(z)$.

(e) Hermite interpolation with points $z_1, \ldots, z_l$ and orders $m_1, \ldots, m_l$ with $m_1 + \cdots + m_l = p$ is the problem of finding a degree p polynomial, $g(z)$, so that $g^{(j)}(z_k) = a_{k,j}$ for $0 \leq j < m_k$ and $1 \leq k \leq l$. Show that if there is a unique Hermite interpolating polynomial for any numbers $a_{k,j}$, then the sequences $\vec{u}_{k,j}$ are linearly independent. Hint: One matrix is the transpose of the other.

(f) A theorem of algebra states that if $z_1, \ldots, z_l$ are the roots of the characteristic polynomial (2) and $m_1, \ldots, m_l$ the corresponding multiplicities, then $m_1 + \cdots + m_l = p$. Show that sequences of the form (3) form a basis for V. Use this to write a formula for the general solution of (1) in terms of special solutions of the form (3).

(g) Show that all solutions of (1) are bounded if and only if the roots of the characteristic polynomial (2) have $|z_k| \leq 1$, and the roots with $|z_k| = 1$ are *simple*, which means $m = 1$.

(h) Let A_r be the $p \times p$ matrix so that

$$\begin{pmatrix} u_{r+p} \\ \vdots \\ u_r \end{pmatrix} = A_r \begin{pmatrix} u_p \\ \vdots \\ u_0 \end{pmatrix} .$$

The matrix A_1 is the *companion matrix* of the recurrence relation (1). Show that $A_r = A_1^r$.

(i) Assume all the roots of the characteristic polynomial are simple. Show that for large r, the condition number of A_r has the order of magnitude

$$\kappa(A_r) \sim \left(\frac{\max_k |z_k|}{\min_k |z_k|} \right)^r .$$

2. Suppose $v = v_1, \ldots, v_{n-1}) \in R^{n-1}$. We extend to $w \in R^N$ with $N = 2n-1$ (numbering the components as $w = (w_{-(n-1)}, \ldots, w_0, w_1, \ldots, w_{n-1})$.)

$$\begin{cases} w_k = v_k \text{ for } 1 \leq k \leq n-1. \\ w_{-k} = -v_k \text{ for } 1 \leq k \leq n-1. \\ w_0 = 0 \end{cases}$$

This is the antisymmetric (or odd) extension of v. Show that the DFT of w allows us to write

$$v_k = \sum_{\alpha=1}^{n-1} \widehat{v}_\alpha \sin\left(\frac{\pi \alpha k}{n}\right) , \tag{4}$$

with

$$\sum_{k=1}^{n-1} v_k^2 = C \sum_{\alpha=1}^{n-1} \widehat{v}_\alpha^2$$

Find the constant, C. Find a formula for $\widehat{v}_\alpha$. This is the discrete Fourier sine transform.

3. We want to approximate the function $u(x, y)$ defined on the unit square $0 \leq x \leq 1$ and $0 \leq y \leq 1$. We want to solve the Poisson problem $\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = f(x, y)$ in the unit square with Dirichlet boundary conditions $u = 0$ when $x = 0$, $x = 1$, $y = 0$, or $y = 1$. We choose $\Delta x = 1/n$ and define an approximate numerical solution so that $u_{jk} \approx u(x_j, y_k)$, with $x_j = j\Delta x$, and $y_k = k\Delta y$). Use the three point second order approximations to $\partial_x^2 u$ and $\partial_y^2 u$ to derive the five point discretize Poisson problem

$$\frac{1}{\Delta x^2} \left(u_{j+1,k} + \cdots \right) = f(x_j, y_k) . \tag{5}$$

(Find the exact formula) Approximate the boundary conditionsby setting $u_{0,k} = u_{n,k} = u_{j,0} = u_{j,n} = 0$. Show that (5) is a system of $(n-1)^2$ equations for the $(n-1)^2$ unknowns u_{jk}. This is a discrete Poisson problem.

4. For each k, let $u_k = (u_{1,k}, \ldots, u_{n-1,k}) \in R^{n-1}$, and let $\widehat{u}_\alpha$ be the discrete Fourier sine coefficients. Show that

$$\widehat{u}_{\alpha,k+1} - m(\alpha)\widehat{u}_{\alpha,k} + \widehat{u}_{\alpha,k-1} = \Delta x^2 \widehat{f}_{\alpha,k} \tag{6}$$

Find the formula for the *symbol*, $m(\alpha)$ and show that $m(\alpha) \geq 2$ for all α. Show that for each α, (6) is a system of linear equations that determines the $n-1$ numbers $\widehat{u}_{\alpha,k}$. Show that this system of equations may be solved in $O(n)$ work using the tridiagonal Choleski factorization. Conclude that it is possible to solve the equations (5) in $O(n^2 \log(n))$ work. How much work would it be to solve the system of equations (5) directly using the full Choleski factorization without the discrete Fourier sine series? This is the basis of *fast Poisson solvers*.

5. Let us temporarily drop the α subscript and consider

$$u_{k+1} - m u_k + u_{k-1} = F_k \ , \tag{7}$$

with $m \geq 2$. We want to impose boundary conditions $u_0 = u_n = 0$. The *shooting* method takes $u_0 = 0$ and attempts to find u_1 so that $u_n = 0$. Suppose we set $u_1 = t$. Show that (7) implies that $u_n = a + bt$. Conclude that two computations of u_n with different t values allow us to determine u_1. Show that this gives a different $O(n)$ algorithm for solving (6) and therefore a different $O(n^2 \log(n))$ algorithm for solving (5). Use the methods for solving recurrence relations in problem 1 to show that this shooting method cannot work in double precision arithmetic for $n \geq 100$.

6. Suppose we have a problem with two discretization parameters Δx and Δt. Suppose the error is supposed to be $O(\Delta t^p + \Delta x^q)$ Suppose that there is an asymptotic error expansion in powers of Δt and Δx. Suggest a strategy using Richardson based convergence analysis to confirm the powers p and q from looking at the output of several runs.

7. Let $A = U\Sigma V^*$ be the singular value decomposition of an $n \times m$ matrix, A. Find a first order perturbation theory formula for $\dot{\sigma}_k$ in terms of $\dot{A}$. Use this formula to conclude that if A is well conditioned than the problem of computing singular values is well conditioned. Is the problem of computing singular vectors well conditioned under the same hypotheses? Hint: $A = I$.