The fourth order four stage Runge Kutta method

We are solving $\dot{y}(t) = f(y(t), t)$, which is a system of k first order differential equations. For each t,

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_k(t) \end{pmatrix} \quad .$$

We have an estimate of $y^{(n)} \approx y(t_n)$. We have a time step Δt so that $t_{n+1} = t_n + \Delta t$. This does not imply that Δt is the same from one time step to the next. The four stages are

$$\begin{aligned} k^{(n,1)} &= \Delta t f(y^{(n)}, t_n) \ , & (1) \\ k^{(n,2)} &= \Delta t f(y^{(n)} + \frac{1}{2} k^{(n,1)}, t_n + \frac{\Delta t}{2}) \ , & (2) \\ k^{(n,3)} &= \Delta t f(y^{(n)} + \frac{1}{2} k^{(n,2)}, t_n + \frac{\Delta t}{2}) \ , & (3) \\ k^{(n,4)} &= \Delta t f(y^{(n)} + k^{(n,3)}, t_n + \Delta t) \ . & (4) \end{aligned}$$

To take a time step, we take a combination of the four stages:

$$y^{(n+1)} = y^{(n)} + \frac{1}{6} \left(k^{(n,1)} + 2k^{(n,2)} + 2k^{(n,3)} + k^{(n,4)} \right) \ . \qquad (5)$$

In the computer implementation, we need to keep the four k vectors, but we do need need separate storage for each value of n.