Assignment 2, due March 17

Corrections: [1a clarified (formula (1)]. The F there is not the same as the stacked F above, but they are related. The inequality in part (d) fixed, so that it is A, not R multiplying x.

1. (*This exercise is the stability/consistency convergence proof for ODE*) Suppose we have an ODE $\dot{x} = f(x)$ with initial condition $x(0) = x_0$. The time step is $\Delta t > 0$. The discrete times are $t_n = n\Delta t$. The approximate solution is $x_n \approx x(t_n)$. We will fix a final time $T > 0$ and define

$$M(T) = \max_{t_n < T} \|x_n - x(t_n)\| \ .$$

A "stacked vector" with r lags is

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-r} \end{pmatrix} \ .$$

In this spirit, we could have stacked functions, such as

$$F(X) = \begin{pmatrix} f(x_n) \\ f(x_{n-1}) \\ \vdots \\ f(x_{n-r}) \end{pmatrix} \ .$$

The function defining the ODE is *globally Lipschitz continuous* if there is an L so that, for all x and y,

$$\|f(x) - f(y)\| \le L \|x - y\| \ .$$

[*Locally* Lipschitz is the condition that this inequality holds for all $\|x - x_0\| \le R$ and $\|y - x_0\| \le R$. Most real nonlinear ODEs are locally Lipschitz but not globally. The proof for locally Lipschitz is more complicated but not harder.]

(a) Show that any linear multistep method may be formulated as

$$X_{n+1} = AX_n + \Delta t \widetilde{F}(X_n, \Delta t) \ . \qquad (1)$$

where $F(X, \Delta t)$ is globally Lipschitz in the X variable if f is globally Lipschitz. The $\widetilde{F}$ is derived from f by stacking and manipulating (linear multistep methods) and by composition (Runge Kutta methods). Constructing F from the linear multistep method may involve more than just stacking. The Lipschitz constant L should stay bounded as $\Delta t \to 0$.

(b) Show that any Runge Kutta method may be formulated in the form (1), with F being globally Lipschitz if f is. This does not involve stacking; you may take $X_n = x_n$. But F is defined from f by applying f to f (iteration or function composition). [The convergence proof in this exercise is hard for linear multistep methods and relatively easy for Runge Kutta methods. The hard parts related to linear algebra are only for linear multistep methods.]

(c) Suppose B is a matrix with eigenvalues λ that all satisfy $|\lambda]| < 1$. Show that there is a symmetric positive definite matrix Q so that if $y = Bx$, then

$$\|y\|_Q^2 \leq \|x\|_Q^2 \ .$$

Here, $\|x\|_Q^2 = x^t Q x$ is the (square of the) norm defined by Q. *Hint*: Define Q indirectly as a quadratic form using

$$x^t Q x = \sum_{n=0}^{\infty} \left(B^n x\right)^t \left(B^n x\right) \ .$$

The sum converges because of the eigenvalue condition. The sum for $y = Bx$ differs from the sum for x only by one term.

(d) The method (1) is *zero stable* if the eigenvalues λ of A satisfy the two conditions

$$\begin{array}{ll} (1) & |\lambda| \leq 1 \\ (2) & \text{if } |\lambda| = 1 \ , \ \text{ there is no Jordan block for } \lambda \end{array}$$

Show that if A is zero stable, then there is a symmetric positive definite matrix R with

$$\|Ax\|_R^2 \leq \|x\|_R^2 \ .$$

Hint: if A is zero stable, there is a basis in which A takes the block form

$$A = \begin{pmatrix} D & 0 \\ 0 & B \end{pmatrix}$$

Here, D is diagonal with all eigenvalues on the unit circle, and B has all eigenvalues inside the unit circle.

(e) Show that there are constants C and C' independent of Δt but depending on R and L and the norm in which L is defined so that if

$$X_{n+1} = AX_n + \Delta t F(X_n, \Delta t) + \Delta t\, r_n$$
and
$$Y_{n+1} = AY_n + \Delta t F(Y_n, \Delta t)$$

then

$$\|X_{n+1} - Y_{n+1}\|_Q \leq (1 + C\Delta t)\, \|X_{n+1} - Y_{n+1}\|_Q + C'\Delta t\, \|r_n\| \ . \quad (2)$$

For this you need the fact from linear algebra that any two vector norms in a given dimension are "equivalent". For any other norm $\|x\|$, there is a $\mu > 0$ so that $\mu\,\|x\| \leq \|x\|_Q \leq \mu^{-1}\,\|x\|$.

(f) Let Y_n be the vector you get by stacking the exact solution $x(t_n)$. The simple residual of a time stepping method is defined by

$$\Delta t\, r_n = Y_{n+1} - [AY_n + \Delta t F(Y_n, \Delta t)] \ .$$

A time stepping method is *consistent* to *order* s if $\|r_n\| \leq C\Delta t^s$. Show that if the time stepping method (1) is consistent to order s, and if $X_0 = Y_0$, then

$$M(T) \leq \Delta t^s\, C' \left(e^{CT} - 1 \right) \ .$$

(g) Show that any Runge Kutta method and any Adams method is zero stable.

(h) (*Nothing to hand in for this*) This is the *stability plus consistency implies convergence* argument of Dahlquist and Lax. The stability condition is the eigenvalue condition of part (d). The consistency condition is the bound on the residual in part (f). If a method is stable, then the ultimate error is of the same order (power of Δt) as the residual.

2. Consider a one lag linear multistep method of the form

$$x_{n+1} = ax_n + bx_{n-1} + \Delta t c f(x_n) + \Delta t d f(x_{n-1}) \ .$$

Find the constants a, b, c, and d to give the method the highest possible order of accuracy. Show that the resulting method is not zero stable.

3. Consider the four stage fourth order Runge Kutta method with $\Delta t = 1$ applied to

$$\dot{x} = i\omega x \ .$$

Find the range of ω for which the method is stable.