

## Assignment 5, Fourier stability analysis, boundary layers

### Corrections and explanations

- Exercise 1 has a correction to the main equation, now (3).
- Exercise 4 has equation (6) corrected. The correction is to add something from the source term, as (4) has.
- There is a discussion section at the end that describes boundary layers and what makes them hard to compute numerically.

**One more clarification:** In calculations involving symbols and von Neumann analysis, please try at first to figure it out analytically. If you are unable or if it seems like it would take “all day”, then try to verify numerically by plotting the amplification factor as a function of  $\theta$  over the range  $0 \leq \theta \leq 2\pi$  and seeing whether the max is more or less than one.

This assignment concerns a linear advection diffusion equation in one space dimension and time

$$\partial_t u + a \partial_x u = \mu \partial_x^2 u + S(x) . \quad (1)$$

The source function  $S(x)$  is specified and the goal is to find the corresponding solution  $u(x, t)$ . In computational examples, take

$$S(x) = 1 , \quad 0 < x < L .$$

We normalize the advection velocity to be one (by non-dimensionalization if necessary):

$$a = 1 .$$

Use Dirichlet boundary conditions

$$u(0, t) = 0 , \quad u(L, t) = 0 .$$

Initial data are that  $u$  starts out being identically zero

$$u(x, 0) = 0 , \quad 0 < x < L .$$

You may take  $L = 1$  in computational examples, again by non-dimensionalization if necessary. The diffusion coefficient  $\mu$  will vary. We are particularly interested in the solution when  $\mu$  is small.

Take a uniform mesh with  $n$  internal grid points  $x_j = j\Delta x$  and

$$\Delta x = \frac{L}{n+1}.$$

Suppose  $V$  is a grid function with values  $V_j$  for  $1 \leq j \leq n$ . The formulas below implicitly use the ghost cell values  $V_0 = 0$  and  $V_{n+1} = 0$ . We define the centered first and second order difference operators as

$$\begin{aligned}(D_0 V)_j &= \frac{1}{2\Delta x} (V_{j+1} - V_{j-1}) \\ (D_+ D_- V)_j &= \frac{1}{\Delta x^2} (V_{j+1} - 2V_j + V_{j-1})\end{aligned}$$

A *Fourier mode* is a grid function  $V_j(\theta) = e^{i\theta j}$ . Be aware that Fourier modes do not satisfy the given Dirichlet boundary conditions. If  $M$  is any translation invariant operator, the corresponding symbol  $m(\theta)$  is defined by

$$MV(\theta) = m(\theta)V(\theta).$$

A semi-discrete approximation to (1) involves a time-dependent grid function  $U_j(t)$

$$\dot{U} = -D_0 U + \mu D_+ D_- U + S. \quad (2)$$

**Exercise 1.** Suppose that  $S$  is the grid function  $S_j = S(x_j)$  and  $u(x, t)$  is a smooth function of<sup>1</sup>  $x$  and  $t$ , and  $U_j(0) = u(x_j, 0)$ . Compute the symbol of the right side of (2). In other words, define the translation invariant operator  $M = -D_0 + \mu D_+ D_-$  and compute  $m(\theta)$ . Use the result to show the semi-discrete scheme is von-Neumann stable. Show that the computed solution is second order accurate in the sense that, for any fixed  $t$ ,

$$\Delta x \sum_{j=1}^n [u(x_j, t) - U_j(t)]^2 = O(\Delta x^4). \quad (3)$$

This represents second order accuracy, for example, if  $u - U = \Delta x^2$  always, then this is satisfied. More systematically, we can define an  $l^2$  norm consistent with the integral  $L^2$  norm

$$\|f\|_{L^2} = \left[ \int_0^L f(x)^2 dx \right]^{\frac{1}{2}} \approx \left[ \Delta x \sum_{0 < x_j < L} f(x_j)^2 \right]^{\frac{1}{2}}.$$

Use forward Euler to make a fully discrete scheme. Define the time  $k$  grid vector  $U_j^{(k)} \approx U_j(t_k)$ , with  $t_k = k\Delta t$ . The scheme is

$$U^{(k+1)} = U^{(k)} + \Delta t [MU^{(k)} + S]. \quad (4)$$

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<sup>1</sup>This is easy to show, assuming the source function is smooth and has the right boundary behavior, using techniques from PDE. If you haven't taken a PDE theory class, you can infer this smoothness from computations below.

Without the advection term, the scheme would be stable under the CFL condition

$$\mu \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}. \quad (5)$$

**Exercise 2.** Code the forward Euler scheme (4). Suppose  $\mu$  is small (say  $\mu = .1$  or  $\mu = .01$ ). Do numerical experiments (plot the numerical solution) on a variety of meshes up to time  $t = \frac{1}{2}$  and several values of  $\Delta x$  to see whether the put diffusion CFL condition (5) determines the stability. Make plots of  $|m(\theta)|$  for  $0 \leq \theta \leq 2\pi$  to see whether the numerical stability/instability you see in the code agrees with the stability prediction based on  $\max |m(\theta)|$ .

**Exercise 3.** Use good  $\Delta x$  and  $\Delta t$  values from Exercise 2 to make a movie of an accurate simulation of (1) up to the time the solution seems to have settled into a steady state. Describe the time dependent and steady state behavior when  $\mu$  is small. There will be a *boundary layer* either near  $x = 0$  or near  $x = L$ . Most of the behavior may be explained using the simplification  $\mu = 0$ , but not the boundary layer.

**Exercise 4.** Consider the partly implicit scheme

$$U^{(k+1)} = U^{(k)} + \Delta t \left[ D_0 U^{(k)} + \mu D_+ D_- U^{(k+1)} + S_j \right]. \quad (6)$$

Show that the scheme is first order accurate if  $\Delta t = \Delta x$  and  $\mu = 1$ . This means showing the residual has the appropriate size and the scheme is stable.

**Exercise 5.** Code a tri-diagonal solver to find  $V$  that satisfies

$$(I - cD_+D_-)V = F.$$

Use this to implement the partly implicit scheme (6). See whether you can do the computation of Exercise 3 faster, first for  $\mu = 1$  and then for smaller values of  $\mu$ .

**Exercise 6.** Try to find time stepping scheme using  $D_0$  and  $D_+D_-$  that is second order accurate and stable with an “advective” time step  $\Delta t = \lambda \Delta x$ . One possibility is a split scheme that does the advective part using an explicit four state Runge Kutta method and the diffusive part using Crank Nicholson. Does this work (is it stable and second order)?

## Discussion

The “physics” of this assignment is *boundary layers*. These are small structures in the solution, often near boundaries, where derivatives (first or second) of the solution are large. The problem (1) with the boundary conditions and source term given will have a boundary layer near one of the endpoints when  $\mu$  is small. The boundary layer is *resolved* if there are enough grid points in it to determine

its structure accurately. Since the boundary layer is small when  $\mu$  is small, this means that  $\Delta x$  must be particularly small.

You can get a mathematical understanding of boundary layers by looking at the derivatives that occur in the PDE. The advective term  $\partial_x u$  involves first derivatives (in space) while the diffusion term involves second derivatives. If first derivatives and second derivatives are comparable and  $\mu$  is small then the second derivative (diffusion) term is small and does not change the solution much. But if the second derivative is much larger than the first derivative, it is possible to have  $\partial_x u$  and  $\mu \partial_x^2 u$  roughly of the same size. The solution to this equation has the second derivative large in this way only near one of the boundary points.

A related mathematical perspective is that if  $\mu$  is small you might explore the consequences of ignoring the diffusion term altogether. In that case, a steady state ( $\partial_t u = 0$ ) cannot satisfy both boundary conditions, so at least one of them will not be satisfied. Putting in the diffusion term must at least have the effect of restoring the boundary condition that was not satisfied without any diffusion term. This requires the diffusion term to change  $u$  by an amount that is not small as  $\mu \rightarrow 0$ . The boundary layer is the thin region where this change happens.

From a physical point of view, the solution “propagates” a distance of order  $\Delta t$  in a time  $\Delta t$ . That is because it moves at some speed, so  $\Delta x$  is proportional to  $\Delta t$ . On the other hand, diffusion can move “stuff” by a distance on the order of  $\sqrt{\Delta t}$  in a time of order  $\Delta t$ . When  $\Delta t$  is small,  $\sqrt{\Delta t}$  is much larger than  $\Delta t$  itself. For example, if  $\Delta t = .01$  then  $\sqrt{\Delta t}$  is .1, which is ten times larger. With a diffusion coefficient  $\mu$ , you get the more dimensionally correct  $\Delta x \sim \sqrt{\mu \Delta t}$ . Consider a spot that is  $\sqrt{\mu \Delta t}$  away from the boundary. Then the diffusion term can “feel” the boundary but the advection term cannot. This suggests that diffusion can have a large effect on the solution, close to the boundary, even if the diffusion coefficient is small.

There are subtle numerical issues in computing boundary layers that you should see and document as you do this assignment. It is possible that the centered difference discretization of the advection term leads to oscillations in the numerical solution that are pure numerical artifact. You can see that oscillations are numerical artifact either by knowing in advance that the PDE solution does not have them, or by doing a convergence study and seeing that the oscillations are not “converged” – they change as  $\Delta t$  and  $\Delta x$  are reduced. In particular, if  $\Delta x$  is too large to “resolve” the boundary layer, then the numerical solution may feature oscillations. Fancier discretizations do not suffer as badly from oscillations.