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Lecture 4: Hyperbolic problems.

<u>wave propagation</u>

sound & accoustics

weather & MHD

electro-magnetic (radar, etc)

eg 1-D accoustics

$\rho(x,t)$ = gas density at location x at time t.

$v(x,t)$ = gas velocity.

<u>Conservation laws</u>: mass & momentum

$M(a,b)$ = mass between a, b

$$= \int_a^b \rho(x,t)\,dx$$

$P(a,b)$ = momentum between a, b

$$= \int_a^b \rho(x,t)\,v(x,t)\,dx$$

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mass density, streaming rate

Mass flux $= F_M = \rho(x,t)\, v(x,t)$

= rate / unit time of mass crossing pt. x at time t.

momentum density

$\underbrace{\text{Momentum flux}}_{} = F_P$

streaming rate

$= \rho(x,t)\, v(x,t) \cdot v(x,t)$

$+ \; p(x,t) \longleftarrow$ gas pressure

$$\frac{d}{dt} M(a,b) = -F_M(b,t) + F_M(a,t)$$

leaving at b

$$\frac{d}{dt} P(a,b) = -F_P(b,t) + F_P(a,t)$$

$$\frac{d}{dt} \int_a^b \rho(x,t)\, dx + \rho(b,t)\, v(b,t) - \rho(a,t)\, \underbrace{v(a,t)}_{=0}$$

$$\cancel{\frac{d}{dt}} \int_a^b \left[\frac{\partial \rho}{\partial t} + \partial_x \rho(x,t)\, v(x,t) \right] dx = 0$$

any a, b.

③

let $a \to b$ & get

$$\partial_t \rho(x,t) + \partial_x\left(\rho(x,t)\, v(x,t)\right) = 0$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + p) = 0$$

Physics $p = p(\rho)$

pressure = fn of density

equation of state, (isentropic)

$$\partial_t \rho + \partial_x(\rho v) = 0$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$

Nonlinear hyperbolic system of equations.

1st picture of what solutions look like - linearize about constant density, $\rho \cong \bar{\rho}$, zero velocity $v = 0$.

small

$$\rho(x,t) = \bar\rho + \tilde\rho(x,t)$$
$$v(x,t) = \tilde v(x,t)$$

$$\rho v = (\bar\rho + \tilde\rho)\,\tilde v = \bar\rho\,\tilde v + \cancel{\tilde\rho\,\tilde v}$$

smaller

$$\rho v^2 = (\bar\rho + \tilde\rho)\,\tilde v^2 = \bar\rho\,\tilde v^2 + \tilde\rho\,\tilde v^2$$

even smaller

$$p(\rho) = p(\bar\rho) + \underbrace{\frac{dp}{d\rho}(\bar\rho)}_{=c^2}\cdot\tilde\rho + \text{even smaller}$$

$$\cancel{\partial_t\bar\rho} + \partial_t\tilde\rho + \partial_x(\bar\rho\,\tilde v) = 0$$

approx

$$\partial_t(\bar\rho\,\tilde v) + \partial_x(c^2\tilde\rho) = 0$$

$$\left.\begin{aligned} \partial_t\tilde\rho + \bar\rho\,\partial_x\tilde v &= 0 \\ \bar\rho\,\partial_t\tilde v + c^2\,\partial_x\tilde\rho &= 0 \end{aligned}\right\}$$ linearized 1-D gas dynamics.

<u>Analysis</u> left and right moving waves, constant amplitude & shape, speed $= \pm s$.

⑤

Seek a solution of the form

$$\begin{pmatrix} \tilde{\rho}(x,t) \\ \tilde{v}(x,t) \end{pmatrix} = \begin{pmatrix} r_\rho \\ r_v \end{pmatrix} A(x-st)$$

- linear, so only the ratio of ~~pres~~ density to velocity disturbance matters. r_ρ, r_v are constants.
- $A(x-st)$ moves to the right with speed s w/o changing shape or size.

(*) becomes

$$r_\rho \cdot (-s) A' + r_v \bar{\rho} A' = 0$$

$$\bar{\rho}(-s) r_v A' + c^2 r_\rho A' = 0$$

Linear algebra formulation:

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$$-sA'\begin{pmatrix} r_\rho \\ r_v \end{pmatrix} + A'\begin{pmatrix} 0 & \bar{\rho} \\ \frac{c^2}{\bar{\rho}} & 0 \end{pmatrix}\begin{pmatrix} r_\rho \\ r_c \end{pmatrix} = 0$$

See that

- $r = \begin{pmatrix} r_\rho \\ r_c \end{pmatrix}$ is an eigenvector
- A' cancels & is irrelevant
  - any wave form propagates

eigenvalue problem

$$s\,r = \begin{pmatrix} 0 & \bar{\rho} \\ \frac{c^2}{\bar{\rho}} & 0 \end{pmatrix} r = 0$$

characteristic polynomial

$$\det\begin{pmatrix} -s & \bar{\rho} \\ \frac{c^2}{\bar{\rho}} & -s \end{pmatrix} = 0$$

$$s^2 - c^2 = 0 \Longrightarrow \boxed{s = \pm c}$$

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$$c = \sqrt{\frac{dp}{d\rho}} = \text{real if } p \text{ is an increasing fn of } \rho.$$

Newton's formula for the speed of sound. Newton measured $\frac{dp}{d\rho}$ in his lab and c outside watching ~~cannon~~ canons. He knew his formula was off by ~ 10% + didn't know why. We now know, Newton measured $\left.\frac{dp}{d\rho}\right|_T$, not $\left.\frac{dp}{d\rho}\right|_S$

constant temperature

constant entropy.

More abstract general analysis - 1D.

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_n(x,t) \end{pmatrix}$$

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we have $u(x,t) = \begin{matrix} \tilde{\rho}(x,t) \\ v(x,t) \end{matrix}$

$A = n \times n$ matrix

$\partial_t u + \partial_x A u = 0$

we have $A = \begin{pmatrix} 0 & \bar{\rho} \\ \frac{c^2}{\bar{\rho}} & 0 \end{pmatrix}$

eigen vectors

$A r_j = s_j r_j \quad j = 1, \ldots, n$

we have

$s_1 = -\sqrt{\frac{dp}{d\rho}} = -c, \quad s_2 = \sqrt{\frac{dp}{d\rho}} = c$

$r_1 = \begin{pmatrix} -\bar{\rho} \\ c \end{pmatrix} \qquad r_2 = \begin{pmatrix} \bar{\rho} \\ c \end{pmatrix}$

check: $A r_1 = \begin{pmatrix} 0 & \bar{\rho} \\ \frac{c^2}{\bar{\rho}} & 0 \end{pmatrix} \begin{pmatrix} -\bar{\rho} \\ c \end{pmatrix}$

$= \begin{pmatrix} c\bar{\rho} \\ -c^2 \end{pmatrix} = -c \begin{pmatrix} -\bar{\rho} \\ c \end{pmatrix} = -c\, r_1$

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$$u(x,t) = \underset{\text{eigenvectors}}{r_1} w_1(x,t) + \cdots + r_n \underset{\text{expansion coefficients}}{w_n(x,t)}$$

Find coefficients using left eigenvector matrix.

$$LR = I, \quad RL = I$$

$$A r_j = s_j r_j \quad \text{or}$$

$$A \begin{pmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ s_1 r_1 & \cdots & s_n r_n \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{pmatrix} \underbrace{\begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}}_{S}$$

$AR = RS \leftarrow$ diagonal speeds matrix

l_k = row k of L

$$L = \begin{pmatrix} - l_1 - \\ \vdots \\ - l_n - \end{pmatrix}$$

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$$L A \underbrace{R L}_{I} = \underbrace{L R}_{I} S L$$

$$L A = S L$$

$$\begin{pmatrix} - l_1 - \\ \vdots \\ - l_n - \end{pmatrix} A = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix} \begin{pmatrix} - l_1 - \\ \vdots \\ - l_n - \end{pmatrix}$$

$$= \begin{pmatrix} - s_1 l_1 - \\ \vdots \\ - s_n l_n - \end{pmatrix}$$

$$l_k A = s_k A :$$

$L =$ matrix of left eigenvectors.

Expansion coefficients

$$\partial_t u + A \partial_x u = 0$$

$$\partial_t L u + L A \partial_x u = 0$$

$$\partial_t w + S \partial_x L u = 0$$

$$\partial_t w + \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix} \partial_x w = 0$$

$$\partial_t w_k + s_k \partial_x w_k = 0$$

Conclusion: in the eigenvector basis, the expansion coefficiente "move" with speed s_j to the right (= left if $s_j < 0$) w/o changing shape. The system of equations $\partial_t u + A \partial_x u = 0$ is equivalent to n independent scalar equation

$$\partial_t w + s \partial_x w = 0$$

"the Kreiss equation".

A 1-D system is strictly hyperbolic if A has n real distinct eigenvalues (speeds). The system is strongly hyperbolic if A is diagonalizable with real eigenvalues, need not be distinct

e.g. Maxwell has 6 components (or 4 if you are more careful) but only 2 distinct wave speeds $s = \pm$ (speed of light).

For accoustics

$$R = \begin{pmatrix} -\bar{\rho} & \bar{\rho} \\ c & c \end{pmatrix}$$

$$\det(R) = -2\bar{\rho}c \qquad \begin{pmatrix} c & -\bar{\rho} \\ -c & -\bar{\rho} \end{pmatrix}$$

$$L = R^{-1} = \frac{-1}{2\bar{\rho}c}\cancel{\begin{pmatrix} c & c \\ \bar{\rho} & -\bar{\rho} \end{pmatrix}} \qquad \text{(check)}$$

$$l_1 = -\frac{1}{2\bar{\rho}c}(c, -\bar{\rho}) = \cancel{\frac{-1}{2\bar{\rho}}(1, 1)} \; -\frac{1}{2}\left(\frac{1}{\bar{\rho}}, -\frac{1}{c}\right)$$

$$\ell_2 = \frac{1}{2}\left(\frac{1}{\bar{\rho}}, \frac{1}{c}\right)$$

$$w_1(x,t) = \frac{1}{2}\left(-\frac{\tilde{\rho}(x,t)}{\bar{\rho}} + \frac{v(x,t)}{c}\right)$$

$$w_2(x,t) = \frac{1}{2}\left(\frac{\tilde{\rho}(x,t)}{\bar{\rho}} + \frac{v(x,t)}{c}\right).$$

- For more complicated problems - Maxwell's eqns, MHD, elastic waves - the algebra is more complicated, but it's worth it!
- Wave propagation for D>1 (multi-D) is more complicated. There is no explicit solution this simple. We study 1-D because
(a) if it doesn't work in 1-D, it won't work in multi-D.

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(b) You can see lots of the things that can go wrong in multi-D already in 1-D

1-D numerics We start with the most natural method and explain why it doesn't work - it's unstable. Then we give some methods that do work.

The bad method: Centered 2nd order in space, forward Euler in time:

$$\partial_x u \cong \frac{u(x+\Delta x, t) - u(x-\Delta x, t)}{2\,\Delta x}$$

$$\partial_t u \cong \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}$$

$$U_{k+1,j} = U_{k,j} - \frac{\Delta t}{2\,\Delta x} A \left(U_{k,j+1} - U_{k,j-1} \right)$$

Program this & you will not see wave propagation. You will see instability.

Why? (Analysis):

Step 1: reduce to the scalar "Kreiss eqn"

$$\cancel{EU} \quad U_{k,j} = \sum_{i=1}^{n} r_i w_{i,k,j}$$

w_{ikj} = amplitude in mode i at time t_k at location x_j

$$LU = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$ (as before

$$LU_{kj} = \begin{pmatrix} w_{1kj} \\ \vdots \\ w_{nkj} \end{pmatrix}$$

algebra as before (LA = SL)

$$LU_{k+1,j} = LU_{k,j} + \frac{\Delta t}{2\Delta x} SL\left(U_{k,j+1} - U_{k,j-1}\right)$$

get

$$w_{i,k+1,j} = w_{i,k,j} + \frac{\Delta t}{2\Delta x} \cdot s_i \cdot \left(w_{i,k,j+1} - w_{i,k,j-1}\right)$$

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Applying the scheme to the hyperbolic system (the u-form with A) is equivalent to applying the scheme independently to each scalar mode component equation

$$\partial_t w + s \partial_x w = 0$$

$$\cancel{w_{k+1} = w_k - \frac{s \Delta t}{2 \Delta x}}$$

$$w_{k+1\,j} = w_{kj} - \frac{s \Delta t}{2 \Delta x} \left(w_{k\,j+1} - w_{k\,j-1} \right)$$

$\lambda = \frac{s \Delta t}{\Delta x}$ = dimensionless measure of time step size = CFL number.

(**) $$\boxed{w_{k+1\,j} = w_{kj} - \frac{\lambda}{2} \left(w_{k,j+1} - w_{k,j-1} \right)}$$

Is this stable? von Neumann analysis. The symbol calculation ~~was~~ uses: left shift $w_{j+1} \rightarrow w_j$ gives

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$e^{i\theta}$, right shift gives $e^{-i\theta}$

(review this form of symbol calculation)

$$a(\theta) = 1 - \frac{\lambda}{2}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$= 1 - i\lambda \; \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$a(\theta) = 1 - i\lambda \sin(\theta)$$

$$\hat{w}_{k+1}(\theta) = a(\theta)\, \hat{w}_k(\theta)$$

to have $\|w_{k+1}\|_2 \leq \|w_k\|_2$

you need

$$\max_\theta |a(\theta)| \leq 1.$$

Calculate:

$$|a(\theta)| = \sqrt{1^2 + \lambda^2 \sin^2(\theta)}$$

For any $\lambda > 0$, $\max_\theta |a(\lambda)| > 1$

$\Rightarrow$ for any λ the method is unstable.

Fixes

1) Upwind differencing in space
2) Lax Wendroff
3) Fancier time-step (Runge Kutta, e.g. see Lecture 5)

1) The "Kreiss eqn" $\partial_t w + s \partial_x w = 0$ is the equation for the evolution of a "passive scalar" carried by "wind" moving at constant speed s. If $s > 0$ the wind is blowing to the right. The "upwind" direction is to the left.

Upwind differencing means using one sided difference approximations to $\partial_x w$ that "look" upwind:

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The simplest is 2 pt, 1st order

$$\partial_x w(x,t) \cong \frac{w(x,t) - w(x-\Delta x, t)}{\Delta x}$$

(if $s > 0$)

$$\partial_x w(x,t) \cong \frac{w(x+\Delta x, t) - w(x,t)}{\Delta x}$$

(if $s < 0$)

The scheme for $\partial_t w + s\partial_x w = 0$ is

$$\frac{W_{k+1,j} - W_{k,j}}{\Delta t} + s\,\frac{W_{k,j} - W_{k,j-1}}{\Delta x} = 0$$

(if $s > 0$)

$$W_{k+1,j} = W_{k,j} - \frac{s\,\Delta t}{\Delta x}\left(W_{k,j} - W_{k,j-1}\right)$$

$$W_{k+1,j} = (1-\lambda)\,W_{k,j} + \lambda\,W_{k,j-1}$$

This is "obviously" stable if $0 < \lambda \le 1$ because W_{k+1} is a convex combination of shifts of W_k. We also could do von Neumann calculations:

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The symbol is

$$a(\theta) = 1 - \lambda(1 - e^{-i\theta})$$
$$= (1-\lambda) + \lambda e^{-i\theta}$$
$$= (1-\lambda) + \lambda(\cos(\theta) + i\sin\theta)$$
$$= 1 - \lambda(1 - \cos(\theta)) + i\lambda\sin(\theta)$$
$$|a|^2 = (1 - \lambda(1-\cos(\theta)))^2 + \lambda^2\sin^2\theta$$
$$\leq 1 \quad \text{for all } \theta \text{ if } 0 \leq \lambda \leq 1$$

ξ

The algebra is complicated but it is not hard.

Conclusion: Upwind differencing is stable if $\Delta t \leq s \cdot \Delta x$.

Drawback: The original problem had wave speeds $s_1 = -c$, $s_2 = +c$

The upwind direction is different for each wave mode. This means you have to do eigenvalue analysis or some other fancy thing to apply upwind differencing to linear acoustics.

Lesson: model problems like the Kreiss equation can be very useful, but you must remember that methods can apply to the model problem but not to the "real" problem.

Lax Wendroff: This is a second order accurate method derived using Taylor series in time.

$$w(x,t+\Delta t) \cong w(x,t) + \Delta t\, \partial_t w(x,t) + \frac{\Delta t^2}{2} \partial_t^2 w(x,t).$$

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From the PDE, we find

$$\partial_t w = - s \partial_x w$$

$$\partial_t^2 w = \partial_t(-s\partial_x w)$$

$$= -s \partial_x \partial_t w$$

$$= -s\partial_x(-s \partial_x w)$$

$$\partial_t^2 w = s^2 \partial_x^2 w .$$

We use centered 2nd order accurate difference approximations to $\partial_x w$ and $\partial_x^2 w$. The result is

$$w_{k+1,j} = w_{k,j} - \Delta t \cdot \frac{s}{2\Delta x}\left(w_{k,j+1} - w_{k,j-1}\right) + \frac{1}{2}\frac{\Delta t^2 s^2}{\Delta x^2}\left(w_{k,j+1} - 2 w_{k,j} + w_{k,j-1}\right)$$

With $\lambda = \frac{s \Delta t}{\Delta x}$, this is

$$w_{k+1,j} = w_{k,j} - \frac{\lambda}{2}\left(w_{k,j+1} - w_{k,j-1}\right) + \frac{\lambda^2}{2}\left(w_{k,j+1} - 2 w_{k,j} + w_{k,j-1}\right).$$

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For a system of linear equations, this would be

$$U_{k+1\,j} = U_{kj} - \frac{\Delta t\, A}{2\Delta x}\left(U_{k\,j+1} - U_{k\,j-1}\right)$$

$$\cancel{+\frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2 U_{jk}}$$

$$+\frac{1}{2}\frac{\Delta t^2}{\Delta x^2} A^2\left(U_{k\,j+1} - 2U_{kj} + U_{k\,j-1}\right).$$

(i.e. this does apply to "real" problems, at least to linear ones).

The symbol is (we've done all this algebra before)

$$a(\theta) = 1 - i\lambda\sin(\theta) + \lambda^2(\cos(\theta) - 1)$$

$$|a(\theta)|^2 = \quad \cdots \quad \leq 1 \text{ if } \lambda \leq 1.$$

By calculation, LW is stable.