Numerical Methods II, Courant Institute, Spring 2016

http://www.math.nyu.edu/faculty/goodman/teaching/NumMethII2016/index.html

Always check the `classes` message board before doing any work on the assignment.

Assignment 3, due March ??

Check Wednesday for a revised version with Problem 5 added

1. (*An exercise in von Neumann analysis*) Consider solving the diffusion equation
$$\partial_t u = D\partial_x^2 u$$
using the three point centered difference approximation to ∂_x^2 and second order explicit two stage Runge Kutta in time. What is the maximum value of the CFL parameter
$$\lambda = \frac{D\Delta t}{\Delta x^2}$$
given by von Neumann stability analysis. Hint: Show that all second order two stage Runge Kutta methods are the same (in exact arithmetic) for linear problems.

2. Find a third order three stage explicit Runge Kutta method. Find the equations that need to be satisfied for such a method to be third order. Make sure your analysis works for general multi-variate nonlinear ODE. Find one or more solutions.

3. For solving the initial value problem for $\partial_t u + A\partial_x u = 0$, we saw that centered second order differencing in space with forward Euler in time is unstable. That scheme is
$$U_{k+1,j} = U_{kj} - \frac{\Delta t}{2\Delta x} A\left(U_{k,j+1} - U_{k,j-1}\right) \ .$$
Friedrichs (the "F" of CFL) suggested replacing U_{kj} on the right by
$$\overline{U}_{kj} = \frac{1}{2}\left(U_{k,j+1} + U_{k,j-1}\right) \ .$$
   (a) Suppose we let $\Delta t \to 0$ and $\Delta x \to 0$ with $r = \frac{\Delta t}{\Delta x}$ fixed. (Warning: r is not dimensionless and is not the CFL number.)

   (b) Show that the Friedrichs scheme is consistent and find its order of accuracy (hint: not great).

   (c) Show that this is not a "method of lines" scheme by showing that the method gives the wrong answer if the time step is too small relative to the space step. Specifically, assume that $\Delta t = \Delta x^2$ (forget units) and show that as $\Delta x \to 0$ the scheme is consistent with a different PDE, one that has a second derivative term in space.

(d) Compute the symbol and use von Neumann analysis to show that the Friedrichs scheme is stable for the "Kreiss equation" $\partial_t w + s\partial_x w$ as long as $\left|\frac{r}{s}\right| \leq 1$.

(e) (Harder) (The argument of Friedrichs, probably) Show directly that if A is symmetric and $r \leq \|A\|$ then

$$\|U_{k+1}\|_2^2 = \sum_j \|U_{k+1,j}\|_2^2 = \sum_j U_{k+1,j}^t U_{k+1,j} \leq \|U_{k+1}\|_2^2 \ .$$

Hint: Use a vector identity like $a_{k+1}^2 - a_k^2 = (a_{k+1} - a_k)(a_{k+1} + a_k)$.

4. (Conservation form of Lax Wendroff). We saw (asserted) that many hyperbolic evolution problems may be formulated as *systems of conservation laws* in the form

$$\partial_t u + \partial_x f(u) = 0 \ . \tag{1}$$

For example we gave the model of a gas (compressible, inviscid) as

$$u(x,t) = \begin{pmatrix} \rho(x,t) \\ \rho(x,t)v(x,t) \end{pmatrix} \quad , \quad f(u) = f(\rho, v) = \begin{pmatrix} \rho v \\ \rho v^2 + p(\rho) \end{pmatrix} \ . \tag{2}$$

We now treat the discrete values U_{kj} not as *point values*, but *cell averages*

$$U_{kj} \approx \frac{1}{\Delta x} \int_{x_j - \frac{1}{2}\Delta x}^{x_j + \frac{1}{2}\Delta x} u(x, t_k)\, dx \ .$$

We saw in an earlier assignment that cell averages are second order accurate approximations to the values at the cell centers. The conservation equation leads to the relation

$$\begin{aligned} \int_{x_j - \frac{1}{2}\Delta x}^{x_j + \frac{1}{2}\Delta x} u(x, t_{k+1})\, dx &= \int_{x_j - \frac{1}{2}\Delta x}^{x_j + \frac{1}{2}\Delta x} u(x, t_k)\, dx \\ &\quad - \int_{t_k}^{t_{k+1}} f(u(x_k + \tfrac{1}{2}\Delta x))\, dt + \int_{t_k}^{t_{k+1}} f(u(x_k - \tfrac{1}{2}\Delta x))\, dt \end{aligned}$$

A *conservation form* scheme takes the form

$$U_{k+1,j} = U_{kj} - \Delta t \left(F_{k,j+\frac{1}{2}} + F_{k,j-\frac{1}{2}} \right) \ .$$

Conservation form schemes differ in their approximations to the *numerical fluxes* $F_{k,j+\frac{1}{2}}$. One such scheme is the Lax Wendroff two-step predictor/corrector method. This is motivated by the (locally third order, globally second order) midpoint rule for the flux integrals

$$\int_{t_k}^{t_{k+1}} f(u(x_k + \tfrac{1}{2}\Delta x))\, dt = \Delta t f(u(x_k + \tfrac{1}{2}\Delta x, t_k + \tfrac{1}{2}\Delta t)) + O(\Delta t^3) \ .$$

The scheme *predicts* the midpoint values using the Lax Friedrichs method

$$\widetilde{U}_{k+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\left(U_{k,j+1} + U_{kj}\right) - \frac{\Delta t}{2\Delta x}\left[f(U_{k,j+1}) - f(U_{kj})\right] \; . \qquad (3)$$

The *corrector* step is

$$U_{k+1,j} = U_{k,j} - \Delta t\left[F\left(\widetilde{U}_{k+\frac{1}{2},j+\frac{1}{2}}\right) - F\left(\widetilde{U}_{k+\frac{1}{2},j-\frac{1}{2}}\right)\right] \; . \qquad (4)$$

(a) Suppose that $u(x,t)$ is a small perturbation of a constant $\overline{u}$. Write this as $u(x,t) = \overline{u} + \dot{u}(x,t)$. Show that, to leading order, $\dot{u}$ satisfies the *linearized* equation

$$\partial_t \dot{u} + A\partial_x \dot{u} = 0 \; ,$$

where $A = f'(\overline{u})$. If u has n components, then A is the $n \times n$ Jacobian matrix of first derivatives of the components of f with respect to the components of u.

(b) For the simplified (no entropy) gas dynamics system (2), the conserved quantities are (mass) density $u_1(x,t) = \rho(x,t)$, and momentum density $u_2(x,t) = \rho(x,t)v(x,t)$. Calculate the 2×2 matrix $A = f'(\overline{\rho}, 0)$ for this case and show that this system is equivalent to the equation we had in class (same wave speeds).

(c) Show that the predictor step (3) is the same as applying the Friedrichs scheme on a grid of size $\frac{1}{2}\Delta x$ for a time step of size $\frac{1}{2}\Delta t$ to the conservation law formulation (1).

(d) Show that for linear problems, $f(u) = Au$, this two step scheme is the same as the Lax Wendroff scheme we saw last week.