Numerical Methods II

Wave propagation equations

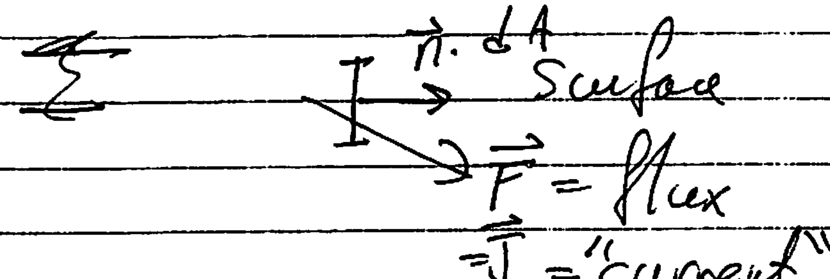
Compressible gas. (incomplete physics).

$\rho(x,t) =$ density

$u(x,t) =$ velocity

$p(x,t) =$ pressure $= p(\rho(x,t))$

equation of state = incomplete.



rate $= \vec{F}\cdot\vec{n}\, dA$

mass flux $= \rho\vec{u}$

momentum flux $= \rho\vec{u}\otimes\vec{u} + p\,I$

(matrix)

x-momentum flux $= \begin{pmatrix} \rho u_x u_x + p \\ \rho u_x u_y \\ \rho u_x u_z \end{pmatrix}$

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y-momentum flux = $\begin{pmatrix} \rho u_x u_y \\ \rho u_y u_y + p \\ \rho u_y u_z \end{pmatrix}$ etc.

Conservation laws

mass: $\partial_t \rho + \nabla \cdot \rho u = 0$

momentum $\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u + p) = 0$

$$\partial_t(\rho u_x) + \partial_x(\rho u_x u_x + p)$$
$$+ \partial_y(\rho u_x u_y) + \partial_z(\rho u_x u_z) = 0$$

$$\partial_t(\rho u_y) + \partial_x(\rho u_x u_y)$$
$$+ \partial_y(\rho u_y u_y + p) + \partial_z(\rho u_y u_z) = 0$$

etc.

in 3D: 4 unknown fields $\rho, u_x, u_y, u_z,$

4 dynamical equations. | not p,

Initial value problem: | $p = p(\rho)$.

given $\rho(x, 0)$, $u(x, 0)$

determine $\rho(x, t)$, $u(x, t)$ $t > 0$

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nonlinear, shock waves - hard.

small disturbance:

$$\rho(x,t) = \bar{\rho} + \dot{\rho}(x,t)$$ small density fluctuation

fixed constant

$$u(x,t) = \dot{u}(x,t) \qquad (\bar{u} = 0)$$

$$\rho u = (\bar{\rho} + \dot{\rho})\,\dot{u} = \bar{\rho}\,\dot{u} + \text{smaller}$$

$$\rho u_x u_x = (\bar{\rho} + \dot{\rho})\,\dot{u}_x \dot{u}_x = \text{smaller}$$

$$p(\rho) = p(\bar{\rho}) + \frac{dp}{d\rho}\cdot\dot{\rho} + \text{smaller}$$

Get

$$\left.\begin{aligned}
&\partial_t \dot{\rho} + \bar{\rho}\,\nabla\cdot\dot{u} = 0 \\
&\bar{\rho}\,\partial_t \dot{u}_x + \frac{dp}{d\rho}\cdot\partial_x \dot{\rho} = 0 \\
&\bar{\rho}\,\partial_t \dot{u}_y + \frac{dp}{d\rho}\cdot\partial_y \dot{\rho} = 0 \\
&\bar{\rho}\,\partial_t \dot{u}_z + \frac{dp}{d\rho}\cdot\partial_z \dot{\rho} = 0
\end{aligned}\right\}$$ linearized ~~acoustics.~~ gas dynamics.

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$$\partial_t \dot{\rho} + \bar{\rho}\, \nabla \dot{u} = 0$$

$$\bar{\rho}\, \partial_t \dot{u} + \frac{dp}{d\rho} \nabla \dot{\rho} = 0$$

Simplify, eliminate $\dot{u}$

$$\partial_t \partial_t \dot{\rho} + \underline{\bar{\rho}\, \nabla \cdot \partial_t \dot{u}} = 0$$

$$\underline{\dot{\rho}\, \partial_t \nabla \dot{u}} + \frac{dp}{d\rho} \nabla \cdot \nabla \dot{\rho} = 0$$

(*) $$\underline{\partial_t^2 \dot{\rho} = \frac{dp}{d\rho} \Delta \dot{\rho}}$$ the wave equation

Simple plane wave solutions

$$\dot{\rho}(\vec{x}, t) = f(x - ct)$$

$$\partial_t^2 \dot{\rho} = c^2 f''(x - ct)$$

$$\Delta \dot{\rho} = f''(x - ct)$$

get from (*):

$$\underline{c^2 f''(x-ct) = \frac{dp}{d\rho} \cdot f''(x-ct)}$$

conclude (1) simple plane waves of any

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shape are possible: $f(x) = $ slope $=$ arbitrary.

(2) waves move with speed $c = \sqrt{\frac{dp}{d\rho}}$.

– Newton formula

(real physics: $\left.\frac{dp}{d\rho}\right|_{\text{temp fixed}}$ (Newton's choice)

or $\left.\frac{dp}{d\rho}\right|_{\text{entropy fixed}}$ – adiabatic

correct

<u>units</u> $[p] = \frac{[\text{force}]}{[\text{area}]} = \frac{ML/T^2}{L^2}$

$= \frac{M}{LT^2}$

$[\rho] = \frac{[\text{mass}]}{[\text{volume}]} = \frac{M}{L^3}$

$\left[\frac{dp}{d\rho}\right] = \frac{[p]}{[\rho]} = \frac{\frac{M}{LT^2}}{\frac{M}{L^3}} = \frac{L^2}{T^2} = \text{velocity}^2$.

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Plane waves in any direction

$$\vec{e} = (e_x, e_y, e_z) \qquad e_x^2 + e_y^2 + e_z^2 = 1$$

$$\dot{\rho} = f(e \cdot x - ct) \quad \text{has}$$

$$\Delta \dot{\rho} = \partial_x^2 f(e \cdot x - ct) + \partial_y^2 f(\ ) + \dots$$

$$= e_x^2 f''(\ ) + e_y^2 f''(\ ) + \dots$$

$$= |e|^2 f'' = f''.$$

any $|e| = 1$, f allowed

<u>isotropic</u> propagation: the same in any direction.

More generally: if Q is an orthogonal matrix and $\dot{\rho}(x,t)$ satisfies (*) then $\dot{\rho}(Qx, t)$ also satisfies (*).

(same calculation, more or less).

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Other solutions - John PDE book

spherical means formula in 3D

$c^2 = \frac{dp}{d\rho}$: accoustic wave equation

Gas dynamics non-accoustic modes

1-D analysis - linear

$(u_x, u_y, u_z) \leftarrow u_x = u$ (1-D)

$(x, y, z) \leftarrow x$ (1-D)

$\rho(x,t) = \bar{\rho} + \dot{\rho}(x,t)$

$u(x,t) = \bar{u} + \dot{u}(x,t)$ $\quad \bar{u} \neq 0$ this time

$\partial_t \rho + \partial_x (\rho u) = 0$

$\partial_t (\bar{\rho} + \dot{\rho}) + \partial_x (\bar{\rho} + \dot{\rho})(\bar{u} + \dot{u}) = 0$

(**) $\partial_t \dot{\rho} + \bar{\rho}\, \partial_x \dot{u} + \bar{u}\, \partial_x \dot{\rho} = 0$

new term

$\partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) = 0$

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$$\partial_t (\bar\rho+\dot\rho)(\bar u+\dot u)$$
$$+\partial_x \left[(\bar\rho+\dot\rho)(\bar u^2+2\bar u\dot u)+\frac{dp}{d\rho}\dot\rho\right]=0$$

$$\bar\rho\,\partial_t \dot u+\bar u\cdot\partial_t\dot\rho$$
$$+2\bar\rho\bar u\,\partial_x \dot u+\bar u^2\partial_x\dot\rho+\frac{dp}{d\rho}\partial_x\dot\rho=0$$

simplify by subtracting $\bar u\cdot$ (**).

(**) $\cancel{\bar\rho}\,\partial_t\dot u+\cancel{\bar\rho}\,\bar u\,\partial_x\dot u+\frac{1}{\bar\rho}\frac{dp}{d\rho}\cdot\partial_x\dot\rho=0$

$$\partial_t\begin{pmatrix}\dot\rho\\ \dot u\end{pmatrix}+\begin{pmatrix}\bar u & \bar\rho\\ \frac{c^2}{\bar\rho} & \bar u\end{pmatrix}\partial_x\begin{pmatrix}\dot\rho\\ \dot u\end{pmatrix}=0$$

General form

~~$v=\dot\rho$~~ $v(x,t)=\begin{pmatrix}\dot\rho(x,t)\\ \dot u(x,t)\end{pmatrix}$

$$A=\begin{pmatrix}\bar u & \bar\rho\\ \frac{c^2}{\bar\rho} & \bar u\end{pmatrix}$$

$v\in\mathbb{R}^m$ ($m=2$ here)

$$\partial_t v+A\partial_x v=0$$

"plane wave" solutions: ~~$v(x,t)=r\cdot f(x-ct$~~

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$$v(x,t) = r \cdot f(x-st)$$

$r \in \mathbb{R}^n =$ solution mode

$s =$ propagation speed.

get

$$-s r f'(x-st) + A r f'(x-st) = 0$$

choose r with

$$A r = s r,$$

Conclusion

* For each real eigenvector of A there is a propagation mode
* Any wave shape is possible for any real mode.
* The eigenvalue is the propagation speed.

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Definition. $\partial_t u + A\partial_x u$
is strongly hyperbolic if there are m linearly independent eigenvectors (propagation modes) with corresponding real eigenvalues (wave speeds).

Check $A = \begin{pmatrix} \bar{u} & \bar{\rho} \\ \frac{c^2}{\bar{\rho}} & \bar{u} \end{pmatrix}$

$\det(A - sI) = 0$

$$0 = \det \begin{pmatrix} \bar{u} - s & \bar{\rho} \\ \frac{c^2}{\bar{\rho}} & \bar{u} - s \end{pmatrix}$$

$(\bar{u} - s)^2 = c^2$

$\underline{s = \bar{u} \pm c}$ (DUH)

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multi-D $A \to (A_x, A_y, A_z)$

$$\partial_t v + A_x \partial_x v + A_y \partial_y v + A_z \partial_z v = 0$$

$\vec{e} = (e_x, e_y, e_z)$ = propagation direction

$r \in \mathbb{R}^m$ = propagation mode

$$v(x,t) = r \cdot f(e \cdot x - st)$$

moves speed s in the $\vec{e}$ direction

get

$$-sf'r + e_x f' A_x r + e_y f' A_y r + e_z f' A_z r = 0$$

$$sr = (e_x A_x + e_y A_y + e_z A_z) r$$

f shape arbitrary.

wave speed and ~~wave~~ propagation

modes may depend on direction.

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2D linearized gas dynamics.

$\rho(x,y,t)$, $u_x(x,y,t)$, $u_y(x,y,t)$

~~$\partial_t \rho + \partial_x (u_x^2 + p)$~~

mass $\partial_t \rho + \partial_x(\rho u_x) + \partial_y(\rho u_y) = 0$

x-momentum $\partial_t(\rho u_x) + \partial_x(\rho u_x^2 + p) + \partial_y(\rho u_y u_x) = 0$

y-momentum $\partial_t(\rho u_y) + \partial_x(\rho u_x u_y) + \partial_y(\rho u_y^2 + p) = 0$

$$\rho(x,y,t) = \bar{\rho} + \dot{\rho}(x,y,t)$$

$$u_x(x,y,t) = \bar{u}_x + \dot{u}_x(x,y,t)$$

$$u_y = \bar{u}_y + \dot{u}_y$$

(1) $\partial_t \dot{\rho} + \bar{\rho}\partial_x u_x + \bar{u}_x \partial_x \dot{\rho} + \bar{\rho}\partial_y u_y + \bar{u}_y \partial_y \dot{\rho} = 0$

$\bar{\rho}\partial_t \dot{u}_x + \cancel{\bar{u}_x \partial_t \dot{\rho}} + \cancel{\bar{\rho}\bar{u}_x} \; 2\bar{\rho}\bar{u}_x \partial_x \dot{u}_x$

$+ \cancel{\bar{u}_x^2 \partial_x \dot{\rho}} + c^2 \partial_x \dot{\rho}$

$+ \bar{\rho}\bar{u}_y \partial_y \dot{u}_x + \cancel{\bar{\rho}\bar{u}_x \partial_y \dot{u}_y} + \cancel{\bar{u}_x \bar{u}_y \partial_y \dot{\rho}} =$

subtract $\bar{u}_x \cdot$ linearized mass eqn:

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get

(2) $\partial_t \dot{u}_x + \bar{u}_x \partial_x \dot{u}_x + \frac{c^2}{\bar{\rho}} \partial_x \dot{\rho} + \bar{u}_y \partial_y \dot{u}_x = 0$

The y-momentum equation is "obviously":

(3) $\partial_t \dot{u}_y + \bar{u}_x \partial_x \dot{u}_y + \bar{u}_y \partial_y \dot{u}_y + \frac{c^2}{\bar{\rho}} \partial_y \dot{\rho} = 0$

Write as a 1st order linear system

$$V(x,y,t) = \begin{pmatrix} \dot{\rho} \\ \dot{u}_x \\ \dot{u}_y \end{pmatrix}$$

$$\partial_t V + \begin{pmatrix} \bar{u}_x & \bar{\rho} & 0 \\ c^2/\bar{\rho} & \bar{u}_x & 0 \\ 0 & 0 & \bar{u}_x \end{pmatrix} \partial_x \begin{pmatrix} \dot{\rho} \\ \dot{u}_x \\ \dot{u}_y \end{pmatrix} + \begin{pmatrix} \bar{u}_y & 0 & \bar{\rho} \\ 0 & \bar{u}_y & 0 \\ c^2/\bar{\rho} & 0 & \bar{u}_y \end{pmatrix} \partial_y \begin{pmatrix} \dot{\rho} \\ \dot{u}_x \\ \dot{u}_y \end{pmatrix} = 0$$

$$A_x = \begin{pmatrix} \bar{u}_x & \bar{\rho} & 0 \\ c^2/\bar{\rho} & \bar{u}_x & 0 \\ 0 & 0 & \bar{u}_x \end{pmatrix}$$

$$A_y = \begin{pmatrix} \bar{u}_y & 0 & \bar{\rho} \\ 0 & \bar{u}_y & 0 \\ c^2/\bar{\rho} & 0 & \bar{u}_y \end{pmatrix}$$

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The eigenvalue problem with direction

$e = (e_x, e_y)$, $e_x^2 + e_y^2 = 1$, is

$$s\, r = e_x A_x r + e_y A_y r$$

$$\det \begin{pmatrix} -s + e_x \bar{u}_x + e_y \bar{u}_y & e_x \bar{\rho} & e_y \bar{\rho} \\ e_x \frac{c^2}{\bar{\rho}} & -s + e_x \bar{u}_x + e_y \bar{u}_y & 0 \\ e_y \frac{c^2}{\bar{\rho}} & 0 & -s + e_x \bar{u}_x + e_y \bar{u}_y \end{pmatrix} = 0$$

The algebra is possible!

define: $\bar{u}_e = e_x \bar{u}_x + e_y \bar{u}_y = \vec{e} \cdot \vec{\bar{u}}$

= background velocity in the $\vec{e}$ direction.

$$\det \left(\begin{array}{cc|c} \bar{u}_e - s & e_x \bar{\rho} & e_y \bar{\rho} \\ \hline e_x \frac{c^2}{\bar{\rho}} & \bar{u}_e - s & 0 \\ \hline e_y \frac{c^2}{\bar{\rho}} & 0 & \bar{u}_e - s \end{array}\right) = 0$$

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expand in cofactors on the 3rd column (one of many ways to do this)

$$(\bar{u}_e - s)\ \det\begin{pmatrix} \bar{u}_e - s & e_x\bar{\rho} \\ e_x\frac{c^2}{\bar{\rho}} & \bar{u}_e - s \end{pmatrix}$$

$$+\ e_y\bar{\rho}\ \det\begin{pmatrix} e_x\frac{c^2}{\bar{\rho}} & \bar{u}_e - s \\ e_y\frac{c^2}{\bar{\rho}} & 0 \end{pmatrix}$$

$$= (\bar{u}_e - s)\left\{(\bar{u}_e - s)^2 - e_x^2c^2\right\}$$

$$- e_y\bar{\rho}\left(e_y\frac{c^2}{\bar{\rho}}(\bar{u}_e - s)\right)$$

$$= (\bar{u}_e - s)\left\{(\bar{u}_e - s)^2 - \cancel{(e_x^2 + e_y^2)}\,c^2\right\}$$

either

$$\bar{u}_e - s = 0 \Longrightarrow s = \bar{u}_e$$ vortical mode / shear mode

or

$$(\bar{u}_e - s) = \pm c \Longrightarrow s = \bar{u}_e \pm c$$ sound waves

Structure/physics of modes: find r.

speed $s = \bar{u}_e$: $r \to \begin{matrix} r_\rho \\ r_{u_x} \end{matrix}$ $\quad r = \begin{pmatrix} r_\rho \\ r_{u_x} \\ r_{u_y} \end{pmatrix}$

$$\begin{pmatrix} 0 & e_x \bar{\rho} & e_y \bar{\rho} \\ e_x \frac{c^2}{\bar{\rho}} & 0 & 0 \\ e_y \frac{c^2}{\bar{\rho}} & 0 & 0 \end{pmatrix} \begin{pmatrix} r_\rho \\ r_{u_x} \\ r_{u_y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

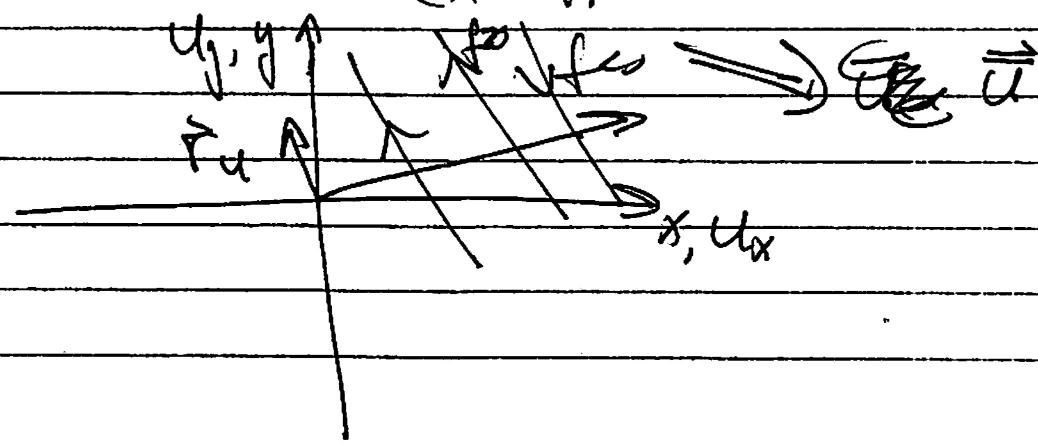
$$e_x \bar{\rho}\, r_{u_x} + e_y \bar{\rho}\, r_{u_x} = 0$$

$$e_x \frac{c^2}{\bar{\rho}}\, r_\rho = 0 \Rightarrow r_\rho = 0 \Rightarrow$$ no density fluctuations.

$$\Rightarrow e_x r_{u_x} + e_y r_{u_y} = 0$$

$$\Rightarrow \vec{e} \cdot \vec{r}_u = 0 :$$

$\underbrace{\text{velocity variation}}_{\vec{r}_u = (r_{u_x},\, r_{u_y})}$ perp. to $\underbrace{\text{prop. dir}}_{\vec{e} = (e_x\, e_y)}$



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Fourier analysis & the initial value problem

Basic idea: orthogonality + completeness.

Take completeness for granted.

Orthogonality for periodic functions

$$f(x+L) = f(x)$$

$$\cancel{\int_0^L e^{-i}} \quad \cancel{\int_0^L e^{2\pi i k}}$$

$$\int_0^L e^{2\pi i n x}\,dx = C\,\delta_n = \begin{cases} C & n=0 \\ 0 & n\neq 0 \end{cases}$$

Therefore $\displaystyle\int_0^L e^{-2\pi i m x} e^{2\pi i n x}\,dx$

$$= \int e^{2\pi i (n-m)x}\,dx = \begin{cases} C & n=m \\ 0 & n\neq m \end{cases}$$

If $f(x) = \sum \hat{f}_n e^{2\pi i n x}$

then

$$\int_0^L e^{-2\pi i m x} f(x)\,dx = \sum \hat{f}_n \int_0^L e^{2\pi i (n-m)x}\,dx$$

$$= C\hat{f}_m$$

Fourier transform

The continuous version of ⊛ is

(⊛⊛) $$\int_{-\infty}^{\infty} e^{ikx}\,dx = 2\pi\delta(k) \quad (\text{Dirac})$$

(Symmetric in k and x)

representation

$$f(x) = \int e^{ikx}\hat{f}(k)\,dk$$

$$\int_{-\infty}^{\infty} e^{-ilx} f(x)\,dx = \iint e^{i(k-l)}\hat{f}(k)\,dk\,dx$$

$$= 2\pi \int \delta(k-l)\hat{f}(k)\,dk$$

$$\int e^{-ilx} f(x)\,dx = 2\pi\,\hat{f}(l)$$

Fourier transform formula

Plancherel thm - follows from orthogonality

$$\int \overline{f(x)}\,f(x)\,dx$$

$$= \iiint e^{-ilx}\overline{\hat{f}(l)}\,e^{ikx}\hat{f}(k)\,dk\,dl\,dx$$

$$= 2\pi \iint \delta(k-l)\,\overline{\hat{f}(l)}\,\hat{f}(k)\,dk\,dl$$

$$= 2\pi \int \overline{\hat{f}(k)}\,\hat{f}(k)\,dk$$

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$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$$

Multi-D. $x \in \mathbb{R}^d$ $\delta(x) = \delta(x_1) \cdots \delta(x_d)$

$$\iint_{\mathbb{R}^d} \cdots \int e^{i \vec{k}\cdot\vec{x}} d\vec{x} = \cancel{\delta(k_1) \cdot c} \; 2\pi\delta(k_1) \cdots 2\pi\delta(k_d)$$

$$= (2\pi)^d \delta(\vec{k})$$

$$\int_{\mathbb{R}^d} |f(x)| dx = (2\pi)^d \int_{\mathbb{R}^d} |\hat{f}(k)| dk.$$

If $f(x)$ is a vector, apply to each component.

<u>Application to initial value problem</u>

$$\partial_t v(x,t) + \sum_{j=1}^{d} A_j \partial_{x_j} v(x,t) = 0$$

$$v(x,0) = \text{given} = v_0(x)$$

write

$$v(x,t) = \int_{\mathbb{R}^d} e^{ik\cdot x} \hat{v}(k,t) dk$$

$$\partial_t v(x,t) = \int_{\mathbb{R}^d} e^{ik\cdot x} \partial_t \hat{v}(k,t) dk$$

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$$\partial_{x_j} v(x,t) = \partial_{x_j} \int e^{ik\cdot x}\, \hat{v}(k)\, dk$$

$$= \int \partial_{x_j} e^{ik\cdot x}\, \hat{v}(k)\, dk$$

$$= \int e^{ik\cdot x}\, i k_j\, \hat{v}(k,t)\, dk.$$

The PDE becomes

$$0 = \int e^{ik\cdot x} \left(\partial_t \hat{v}(k,t) + \left(\sum_{j=1}^{d} i k_j A_j \right) \hat{v}(k,t) \right) dk$$

Which (Fourier uniqueness, Plancherel)

(*) $\partial_t \hat{v}(k,t) = -i\, (k\cdot A)\, \hat{v}(k,t)\, dk$

with $\hat{v}(k,0) = \hat{v}_0(k) =$ given

$$k\cdot A = \sum_{j=1}^{d} k_j A_j = \text{a } d\times d \text{ matrix}$$

that depends on k.

(*) is a d-dim linear ODE system –

$$\vec{k}\cdot\vec{A} = |k| \cdot \vec{e}_k \cdot \vec{A}$$

$$\vec{e}_k = \frac{k}{|k|} = \text{unit vector in the } k \text{ direction}$$

$|k\cdot A| \to \infty$ as $k \to \infty$

(2) $\partial_t \hat{v} = -i\,(k\cdot A)\,\hat{v}$

can have solutions that grow arbitrarily fast as $k \to \infty$, unless they don't.

Strongly hyperbolic ~~eq~~:

$k\cdot A$ has a full family of real eigenvectors (with real eigenvalues DUH) for each $k \in \mathbb{R}^d$.

~~for~~ $(e_k \cdot A)\, r_j = -s_j\, r_j$ $j = 1, \ldots, d$

s_j, r_j real, fnc of e_k, linearly independent.

Homogeneity $s_j(\vec{k}) = s_j(\vec{e}_k)\cdot|k|$

if $s_j(\vec{e}_k) = \sigma + i\tau$ with $\tau \neq 0$

then $\bar{s}_j = \sigma - i\tau$ is also an eigenvalue

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Get solns of the ODE system (*)

$$\hat{v}(k,t) = a \cdot e^{i k \kappa_j + |k| \tau_j} r_j .$$

The growth ~~rate is ab rate blows~~

rate $\to \infty$ as $k \to \infty$.

This is the practical definition of <u>illposed</u>

<u>Conclusions</u>

1) You understand the basics of hyperbolic PDE looking for modes or plane waves

2) You can be led to this using the Fourier transform.

3) The Fourier transform analysis also shows that a problem that fails to be hyperbolic (of the form $\partial_t u \neq \sum A_j \partial_{x_j} u = 0$) is ill posed. Being strongly hyperbolic makes it well posed in L^2.

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Back to 1-d strongly hyperbolic

$$\partial_t v + A \partial_x v = 0 \qquad A = d\times d$$
$$v \in \mathbb{R}^d$$

$$A r_j = s_j r_j$$

row vectors

left eigenvectors $l_j A = s_j l_j$

bi-orthogonality

$$l_j \cdot r_k = \delta_{jk}$$

Eigenvector matrix

$$R = \begin{pmatrix} | & & | \\ r_1 & \cdots & r_d \\ | & & | \end{pmatrix}$$

$$L = \begin{pmatrix} - & l_1 & - \\ & \vdots & \\ - & l_d & - \end{pmatrix}$$

$$AR = A \begin{pmatrix} | & & | \\ r_1 & \cdots & r_d \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ s_1 r_1, & \cdots, & s_d r_d \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ r_1 & \cdots & r_d \\ | & & | \end{pmatrix} \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_d \end{pmatrix}$$

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$$AR = RS$$

where $S = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_d \end{pmatrix}$

= diagonal matrix with wave speeds

Similarly $LA = \begin{pmatrix} - l_1 - \\ \vdots \\ - l_d - \end{pmatrix} A$

$$= \begin{pmatrix} - s_1 l_1 - \\ \vdots \\ - s_d l_d - \end{pmatrix} = SL.$$

Also $LR = \begin{pmatrix} - l_1 - \\ \vdots \\ - l_d - \end{pmatrix} \begin{pmatrix} | & & | \\ r_1 & \cdots & r_d \\ | & & | \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} = I$$

$\underline{L = R^{-1}}$

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$AR = RS$ — eigenvector relations

$LAR = LRS = S$

$A = RSR^{-1} = RSL$

$\partial_t v + A \partial_x v = 0$

$\partial_t Lv + LA \partial_x v = 0$

set $w = Lv$

$\partial_t w + SL \partial_x v = 0$

$\partial_t w + S \partial_x w = 0$

$\partial_t w_j(x,t) + S_j \partial_x w_j = 0$

The $w = Lv$ $\quad Rw = RLv = v$

$$v = \begin{pmatrix} | & & | \\ r_1 & \cdots & r_d \\ | & & | \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix} = r_1 w_1(x,t) + \cdots + r_d w_d(x,t)$$

w_j = amplitude of r_j in a mode expansion

The j mode satisfies

$\partial_t w_j + S_j \partial_x w_j = 0$

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w_j moves with speed s without changing shape. General soln

$$U_0(x) = U(x,0)$$

$$W(x,0) = L\,U(x,0)$$

$$W_j(x,0) = l_j \cdot U(x,0)$$

$$U(x,t) = \sum_{j=1} r_j\, w_j(x - s_j t)$$

where

$$w_j(x) = l_j \cdot U(x,0)$$

Superposition of waves moving with different speeds in different directions

Aside discrete fourier transform

suppose f_j represents fn values at points $x_j = j \cdot h$ ~~$j\in$~~ $j \in \mathbb{Z}$.

The Dirac orthogonality relation is

$$h \cdot \sum_{j=\infty}^{\infty} e^{ikx_j} = 2\pi\delta(k)$$

$$\text{if } \frac{-\pi}{h} < k < \frac{\pi}{h}$$

The "Fourier transform" of the discrete grid fn is

$$\hat{f}(k) = ch \cdot \sum_{j=\infty}^{\infty} e^{-ikx_j} f_j$$

This is periodic in k with period $\frac{2\pi}{h}$. $\hat{f}\left(k + \frac{2\pi}{h}\right) = \hat{f}(k)$

A Fourier representation of f_j is

(*) $$f_j = \int_{-\pi/h}^{\pi/h} e^{ikx_j} \hat{f}(k)\,dk.$$

as before, the Dirac formula and (*) gives

(**) $$h \sum_{j=-\infty}^{\infty} |f_j|^2 = 2\pi \int_{-\pi/h}^{\pi/h} |\hat{f}(k)|^2\,dk$$

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Numerical soln of hyperbolic PDE - 1D.

$x_j = j \cdot \Delta x$, ~~$t_k = k$~~ $t_n = n \cdot \Delta t$

$v_{n,j} \cong v(x_j, t_n)$

→ grid values

$$\partial_t v(x,t) \cong \frac{v(x, t+\Delta t) - v(x,t)}{\Delta t}$$

$$\partial_x v(x,t) \cong \frac{v(x+\Delta x, t) - v(x-\Delta x, t)}{2\Delta x}$$

$$\partial_t v(x_j, t_n) \leftarrow \frac{v_{n+1,j} - v_{n,j}}{\Delta t}$$

$$\partial_x v(x_j, t_n) \leftarrow \frac{v_{n,j+1} - v_{n,j-1}}{2\Delta x}$$

$$\partial_t v(x_j, t_n) + A\, \partial_x v(x_j, t_n) = 0$$

$\Longrightarrow$

bad scheme

$$\frac{v_{n+1,j} - v_{n,j}}{\Delta t} + A\, \frac{v_{n,j+1} - v_{n,j-1}}{2\Delta x} = 0$$

BS $\Longrightarrow$ $v_{n+1,j} = v_{n,j} - \frac{\Delta t}{\Delta x} \cdot \frac{1}{2} A \left(v_{n,j+1} - v_{n,j-1}\right)$

(58)

Marching in time

$v_{0,j}$ given for all j

for $n = 0, \ldots, t_n \le T$

for all j

$$v[n+1][j] = v[n][j] + \frac{\Delta t}{2\Delta x} A \left(v[n][j+1] - v[n][j-1] \right)$$

Forward Euler in time,

centered difference in space.

This does not work. It is unstable.

A fix: Lax, Wendroff

$$v(x, t+\Delta t) = v(x,t) + \Delta t\, \partial_t v(x,t) + \frac{\Delta t^2}{2} \partial_t^2 v(x,t) + O(\Delta t^3)$$

Turn the t derivatives into x derivs using the PDE: $\partial_t v = -A \partial_x v$

$$\partial_t^2 v = \partial_t \partial_t v = -\partial_t A \partial_x v$$

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$$= -A\partial_x \partial_t v$$
$$= A^2 \partial_x^2 v.$$

$$v(x, t+\Delta t) = v(x,t) \cancel{x} \Delta t A \partial_x v(x,t)$$
$$+ \tfrac{1}{2}\Delta t^2 A^2 \partial_x^2 v(x,t) + O(\Delta x^3).$$

Scheme

$\cancel{v_j}$

LW
$$v_{n+1,j} = v_{n,j} - \frac{\Delta t}{2\Delta x} A\left(v_{n,j+1} - v_{n,j-1}\right)$$
$$+ \frac{\Delta t^2}{2\Delta x^2} A^2 \left(v_{n,j+1} - 2v_{n,j} + v_{n,j-1}\right)$$

This does ~~work~~ work

It is stable & converges as $\Delta x, \Delta t \to 0$ in the right way (CFL condition).

Fourier analysis (von Neumann analysis)

Bad scheme: apply L to get a scalar

$$w_{n+1,j} = w_{n,j} - \frac{\Delta t\, S}{2\Delta x}\left(w_{n,j+1} - w_{n,j-1}\right).$$

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$v \to w$ (vector to scalar)

$A \to s$ (matrix to scalar)

Fourier analysis

$$\hat{w}_{n+1}(k) = \hat{w}_n(k) - \frac{\Delta t s}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\hat{w}_n(k)$$

$$\frac{2i}{2i}\,\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{} = 2i\sin(k\Delta x)$$

$$\hat{w}_{n+1}(k) = \underbrace{\left(1 - \frac{i\,\Delta t s}{\Delta x}\sin(k\Delta x)\right)}_{\text{Fourier multiplier}}\hat{w}_n(k)$$

Fourier multiplier

= Symbol of the BS scheme

dimensionless number

λ = CFL number $= \frac{s\,\Delta t}{\Delta x}$

(Courant, Friedrichs, Lewy)

λ = # of grid "zones" per time step moving at speed s.

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Computational practice:

Take $\Delta x \to 0$, $\Delta t \to 0$ with λ fixed

Then $\Delta t = \frac{\lambda \Delta x}{S}$ ~~pro~~

= proportional to Δx.

# of time steps $\to \infty$ as $\Delta x \to 0$.

To solve up to a fixed time T with N points in space takes $O(N^2)$ work.