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Numerical Methods II Spring 2014

Laplace Equation

space variables $(x, y, z) = x = (x_1, x_2, x_3)$

dim $= d = 3$

$u(x) = u(x, y, z)$ = concentration field

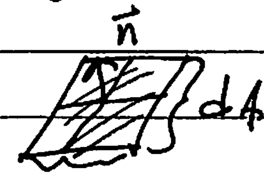
$q(x)$ = steady sources / sinks

$\Delta u = q$

$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = q$

Derivation: Fick's law

$F = (F_x, F_y, F_z)$ = flux



$\text{Flux} = \vec{n} \cdot \vec{F}\, dA = \vec{F} \cdot \vec{dA}$

steady state $\vec{\nabla} \cdot \vec{F} = q$

Fick's law: $\vec{F} = -D \cdot \nabla u$ $(D = 1 \text{ here})$

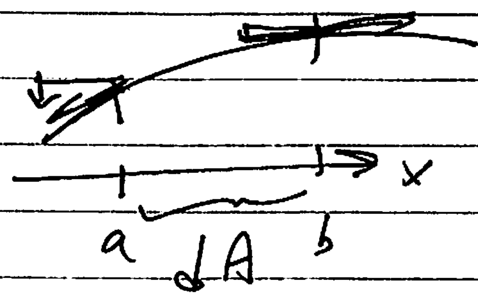
$\boxed{✓ \quad \Delta u = -q.}$

Heat (diffusion) equation

$$\cancel{\text{Eq}}\ \partial_t u + \nabla\cdot F = q$$

$$\boxed{(*)\quad \partial_t u - \Delta u = q}$$

Wave equation



$d=1$

$$\text{Force} = \partial_x u(b,t) - \partial_x u(a,t)$$

$$= \int_a^b \partial_x^2 u(x,t)\,dx$$

$$\text{accel} = \rho\cdot\partial_t^2 u\cdot dA$$

get $\rho\,\partial_t^2 u = \partial_x^2 u$

3-D: $\boxed{\rho\,\partial_t^2 u = \Delta u}$

set $\rho=1$ often.

Ref: basic PDE book: Fritz John,

③

Boundary conditions : part of the model.

A small fraction of the computer work.

The majority of the program & mathematical complexity.

Dirichlet: $u(x) = 0$ on $\partial\Omega$

satisfy PDE inside Ω

e.g. concentration / displacement $= 0$.

Neumann: $\partial_n u = -F_n = 0$ on $\partial\Omega$

no flux / force at bdry.

Periodic $u(x+L, y, z) = u(x, y+L, z)$

$= u(x, y, z+L) = u(x, y, z)$

$L =$ period.

usually to simplify math - avoid boundary conditions. Not often physical.

<u>Fourier modes</u> $\mathbb{R}^d$ $v_k(x) = e^{ik\cdot x} = e^{ik_1x_1 + k_2x_2 + k_3x_3}$.

k = wave vector or d: wave number

wave length (1-d) λ: $e^{ik\lambda} = 1$

$\lambda = \frac{2\pi}{k}$ large wave number $\Leftrightarrow$ small wave length

heat eqn soln $A(t)\, v_k(x)$ $\dot{A} = -k^2 A$

$$u(x,t) = e^{ikx - k^2 t} = e^{-k^2 t}\, e^{ikx}$$ $k^2 = k_1^2 + k_2^2 + k_3^2 = |k|^2$.

short wave length $\Leftrightarrow$ fast decay

wave eqn: $u(x,t) = A(t)\, e^{ikx}$

$$\ddot{A} = -k^2 A: \quad A = e^{-i\omega t} \quad \omega = \pm|k|$$

short wave $\Rightarrow$ fast oscillation.

<u>Fourier modes</u>, period L $v_n(x) = e^{2\pi i n\cdot x/L}$

$n = (n_1, n_2, n_3) \in \mathbb{Z}^d$ ($d = 3$ here)

$n = 0$ = const. $|n| = 1$ = longest wave

$\Rightarrow$ slowest time scale = L^2 or L.

Discrete Fourier modes lattice spacing h

$x_j = (x_{j_1}, x_{j_2}, x_{j_3}) = (j_1 h, j_2 h, j_3 h)$

period L, $Nh = L$

N = # lattice points in each direction

lattice points in all $= N^d$
= big if d is large.

Range of length scales $l_{min} = h$, $l_{max} = L$

$$\cancel{v_k(x) =} \quad \cancel{v_\alpha(x) = e} \quad v_n(x_j) = e^{2\pi i n x_j / L}$$

$$= e^{2\pi i \frac{h}{L} n \cdot j} = e^{i\alpha n \cdot j}$$

aliasing: different $n \not\Rightarrow$ different v_n.

$d=1$: $n' = n + N$ has

$$v_{n'}(x_j) = v_n(x_j) \quad \text{(check)}$$

complete list of distinct modes

$n = 0, 1, \dots, N-1$ or

$n = -\frac{N}{2}+1, \dots, 0, \dots, \frac{N}{2}$ (N even)

$n = -\frac{N-1}{2}, \dots, \frac{N-1}{2}$ (N odd)

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\# of modes = \# of lattice points.

V = vector space of grid functions

"standard" basis $e_j \in V$

$$e_j(x_i) = \begin{cases} \frac{1}{h} & \text{if } i = j \pmod N \\ 0 & \text{if } i \neq j \pmod N. \end{cases}$$

inner product: $u, w \in V$

$$\langle u, w \rangle = \underset{h}{h} \cdot \sum_{x_j \in B} \bar{u}(x_j)\, w(x_j)$$

to be consistent with integration.

$$\langle e_j, e_k \rangle = \delta_{jk}$$

$\dim(V) = N = \#$ lattice points

B = "box" = complete set of distinct lattice points

e.g. x_j, $j = 0, \ldots, N-1$.

$d > 1$: $\langle u, w \rangle = h^d \sum_{x_j \in B} \bar{u}(x_j)\, w(x_j)$

$\dim = N^d$.

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Fourier basis: Fourier modes

$$v_n(x_j) = c\, e^{2\pi i n x_j / L}$$

Complete set of modes $n \in B'$ (dual lattice)

Prop: if $n, m \in B'$, $n \neq m$ then

$$\langle v_n, v_m \rangle = 0.$$

Pf: geometric series.

Normalization $\langle v_n, v_n \rangle = 1$ if

$$h^d \cdot N^d c^2 = 1 \qquad Nh = L$$

$$c = \frac{1}{L^{d/2}}$$

DFT - discrete Fourier Transform

Fourier modes = ortho-normal basis

$u \in V$ has $u = \sum_{n \in B'} \hat{u}_n v_n$

$$\hat{u}_n = \langle v_n, u \rangle = h^d \sum_{x_j \in B} \frac{1}{L^{d/2}} e^{-2\pi i\, n \cdot x_j / L} \cdot u(x_j).$$

$$u(x_j) = \sum_{n \in B'} \hat{u}_n \frac{1}{L^{d/2}} e^{2\pi i\, n \cdot x_j / L}.$$

⑧

$$B' = \begin{cases} \{ -\frac{N}{2}+1, \dots, 0, \dots, \frac{N}{2} \} & \text{if } N \text{ even} \\ \{ -\frac{N+1}{2}, \dots, 0, \dots, \frac{N-1}{2} \} & \text{if } N \text{ odd} \end{cases}$$

Some modes, N even

$n = 0$, $v_0(x_j) = \frac{1}{L^{d/2}} = \text{const}$ all x_j.

$n = n_{max} = N/2$, $v_{N/2}(x_j) = \frac{1}{L^{d/2}} \cdot (-1)^{|j|} \left(= \pm 1 \right)$

n small: long waves, slowly varying fn.

accurate interpolation:

$$v_n(x_{j}) \approx \frac{1}{2}\left(v_n(x_{j+1}) + v_n(x_{j-1}) \right)$$

$|n| \sim N$ short waves, grid scale,

inaccurate interpolation.

Properties of DFT basis

- Planchard formula $\|u\|^2_{\ell^2(B)} = \langle u, u \rangle$

$$= h^d \sum_{x_j \in B} |u(x_j)|^2$$

$$= \|\hat{u}\|^2_{\ell^2(B')} = \sum_{n \in B'} |\hat{u}_n|^2.$$

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- Diagonalizes translation invariant operators.

~~def~~ Right shift on $\mathbb{B}$ moves values to the right one point: if $w = Tu$ then ~~$w_j = u_{j-1}$~~

$w_j = u_{j-1}$ (mod N).

~~an operat~~ an operator (linear, matrix) A is <u>translation invariant</u> if it commutes with translation

$$TAu = ATu$$

e.g. finite difference operator:

$$\partial_x u(x_j) \cong \frac{1}{h}\left(u(x_{j+1}) - u(x_j)\right)$$

$$\partial_x u \cong D_h^+ u$$ (1st order one-sided difference approx).

$$\partial_x^2 u(x_j) \cong \frac{1}{h^2}\left(u(x_{j+1}) - 2u(x_j) + u(x_{j-1})\right)$$ (2nd order centered)

$$\partial_x^2 u \cong D_h^+ D_h^- u$$

<u>Proposition</u>[1] If A is translation invariant then DFT modes are eigenvectors of A.

Proof (abstract version):

Lemma Let ~~E~~ A and T be any commuting operators on vector space V. Let $V_\lambda \subseteq V$ be the λ-eigenvectors of T: ~~$Tu = \lambda u$ for all u~~

$$u \in V_\lambda \iff Tu = \lambda u.$$

Then if $u \in V_\lambda$ and $w = Au$ then $w \in V_\lambda$. This is $A : V_\lambda \to V_\lambda$.

Proof of Lemma: need to show that

$Tw = \lambda w$, but

$$Tw = TAu = ATu = \lambda Au = \lambda w. \quad \text{QED.}$$

For T = right shift, $Tv_n = \lambda_n v_n$ with

$$\lambda_n = e^{-2\pi i \frac{h}{L} n}$$

check: if $n \in B'$, $n' \in B'$, $n \neq n'$, then $\lambda_n \neq \lambda_{n'}$

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Thus: v_n, $n \in B'$, is a complete set of N eigenvectors, with N distinct eigenvalues.

~~If~~ $\lambda = \lambda_n$ then V_{λ_n} is <u>one dimensional</u>

otherwise, $V_\lambda = \{0\}$ = trivial.

Thus, $V_{\lambda_n} = \text{span}(v_n)$, if ~~etc~~

if $w \in V_{\lambda_n}$, then $w = m\, v_n$.

End of abstract proof:

$A v_n \in V_{\lambda_n}$ so there is an

~~$m_A(n)$~~ $m_A(n)$ so that $\boxed{A v_n = m_A(n)\, v_n}$

The function $m_A(n)$ is the <u>symbol</u>

or <u>Fourier multiplier</u> for A. <u>QED</u>

If you know $m_A(n)$, you can compute A

$$Au = A \sum \hat{u}_n v_n = \sum \hat{u}_n A v_n$$

$$\underline{Au = \sum \hat{u}_n\, m_A(n)\, v_n =}$$

Note: non constructive: no formula for the eigenvalue.

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Fourier multiplier: $\hat{u}_n \xrightarrow{A} m_A(n)\,\hat{u}_n$.

e.g. if $A = D_h^+$ then

$A\, e^{2\pi i n x_j/L}$ note $x_{j+1} = x_j + h$

$= \frac{1}{h}\left(e^{2\pi i n (x_j+h)/L} - e^{2\pi i n x_j/L}\right)$

$= \frac{1}{h}\left(e^{2\pi i n h/L} - 1\right) e^{2\pi i n x_j/L}$

so $\boxed{m_{D_h^+}(n) = \frac{1}{h}\left(e^{2\pi i n h/L} - 1\right)}$

Note: $m_{D_h^+}(n) \to \frac{2\pi i n}{L}$ as $h \to 0$, n fixed.

Important formulas

$D_h^0 u = \frac{1}{2h}\left(T^{-1} u \mp T^{1} u\right)$

$D_h^0 u(x_j) = \frac{1}{2h}\left(u(x_{j+1}) - u(x_{j-1})\right)$

$D_h^0 = \frac{1}{2}\left(D_h^+ u + D_h^- u\right)$

$= 2^{nd}$ order centered 1^{st} deriv. approx

$m_{D_h^0}(n) = \cancel{2h}\, i \cdot \frac{\sin(2\pi n h/L)}{h}$

skew symmetric D_h^0 has purely imaginary eigenvalues.

$$D_h^+ D_h^- \to m_{D_h^+ D_h^-}(n) = 2\,\frac{\cos(2\pi nh/L) - 1}{h^2}$$

$$\begin{aligned} m(n) &\le 0 \quad \text{all } n \\ &< 0 \quad \text{if } n \in B',\ n \ne 0. \end{aligned}$$

<u>Thm (von Neumann analysis):</u> if A is translation invariant, and $\|A\| = \sup_{u \ne 0} \frac{\|Au\|_{\ell^2(B)}}{\|u\|_{\ell^2(B)}}$, then

- $\|A\| = \max_{n \in B'} |m_A(n)|$ (1)
- $\|A\| \le \max_{k \in \mathbb{R}} |m_A(k)|$ (2)

<u>Proof:</u> clearly (1) implies (2). (1) is

$$\|Au\|^2_{\ell^2(B)} \le \max_n |m_A(n)|^2 \|u\|^2_{\ell^2(B)}$$

$$\sum |m_A(n)\,\hat{u}_n|^2 \le \left(\max |m_A(n)|^2\right) \cdot \sum |\hat{u}_n|^2.$$

<u>Remark:</u> usually, (2) $\to$ (1) as $h \to 0$ with L fixed

For example, the numbers

$$\frac{2\left(\cos\left(2\pi n \frac{h}{L}\right) - 1\right)}{h^2} = m_{D_h^+ D_h^-}(n)$$

are ~~more~~ increasingly close points on the curve

$$\frac{2}{h^2} \cdot \left(\cos(kh) - 1\right) = m(k)$$

as $k \to 0$, $m(k) \sim -k^2$

The max absolute value is when $\cos(kh) = -1$

$$\max_k |m(k)| = \frac{4}{h^2}. \qquad kh = \pi.$$

If N is even, then $n = \frac{N}{2}$ has

$$m_{D^+D^-}\left(\frac{N}{2}\right) = \frac{2}{h^2}\left(\cos(\pi) - 1\right) = \frac{-4}{h^2}$$

exactly.

If N is odd, $n_{max} = \frac{N-1}{2}$ has

$$2\pi\, n_{max}\, h/L = \frac{\cancel{2}\pi \cancel{N h}}{\cancel{2}\cancel{L}} - \frac{\cancel{2}\pi h}{\cancel{2}L} = \pi - O(h).$$

Proof of Proposition 1, (p. 9) concrete version.

Claim: An operator A is translation invariant if & only if it is a convolution. If a_j for $j \in B$ is a set of numbers and ~~a_j~~ $u \in V$, then

$$\boxed{(a * u)_j = \sum_{k \in B} a_{j-k} u_k}$$

where a_{j-k} means either that a_j is periodic ~~in~~ with period ~~$\dots = a_{j+N}$~~ N, $a_{j+N} = a_j$, or ~~a_{j-k} is mod N.~~ $a_{j-k} = a_{(j-k) \bmod N}$.

Pf of claim: (i) convolutions are translation invariant, (ii) translation invariant $\Rightarrow$ a convolution.

(i) Suppose $u' = Tu$, which means

$u'_j = u_{j-1}$, $w = a * u$, $w' = a * u'$.

We want to show that $w' = Tw$.

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which is $w'_j = w_{j-1}$. Simple algebra:

$$w_{j-1} = \sum_k a_{j-1-k}\, u_k$$

set $k+1 = k'$, $k = k'-1$

$$\sum_{k\in B'} \Leftrightarrow \sum_{k'\in B'} \quad \text{(check)}$$

$$w_{j-1} = \sum a_{j-k'}\, u_{k'-1}$$

$$= (a * u')_j \qquad QED$$

(ii) Define standard basis elements

$$e_k = T^k e_0, \quad e_{0j} = \delta_{0j}$$

$e_k \in V$, form a basis.

$$u = \sum_{k\in B} \underset{\text{number}}{u_k}\, \underset{\text{basis element}}{e_k}$$

$$Au = \sum u_k A e_k$$

$$= \sum u_k A T^k e_0$$

$$Au = \sum u_k T^k A e_0$$

Define $a \in V$ as $a = Ae_0$.

Then $\boxed{Au = \sum_k (T^k a) \cdot u_k}$

With indices, this is

$$(Au)_j = \sum_{k \in B} (T^k a)_j \, u_k$$

$$= \sum_k a_{j-k} \, u_k \qquad \text{QED, (claim).}$$

~~Pf of proposition:~~ ~~$u_k = e^{2\pi i hk/L}$ then~~

~~$(Au)_j = (a * u)_j$~~

~~$=$~~

Claim: convolution is commutative:

$$a * u = u * a$$

Pf: $(a * u)_j = \sum_{k \in B} a_{j-k} \, u_k$

set $k' = j - k$, so $k = j - k'$

$$= \sum_{k' \in B} a_{k'} \, u_{j-k'}$$

$$= (u * a)_j \qquad \text{QED}$$

~~Pf of proposition: $u_k = e^{2\pi i hk/L}$~~

Pf of proposition: v_n has

$$v_{nj} = e^{2\pi i h n j/L}$$

$$\begin{aligned}
(A v_n)_j &= (a * v_n)_j \\
&= (v_n * a)_j \\
&= \sum_k e^{2\pi i h n j/L} \cdot e^{-2\pi i h n k/L} a_k \\
&= m(n) \underbrace{e^{2\pi i h j/L}}_{v_{nj}}
\end{aligned}$$

$$\boxed{m_A(n) = \sum_k a_k e^{-2\pi i h n k/L}}$$

$$= c \cdot \langle v_n, a \rangle$$

Constructive proof gives a formula for the multiplier/eigenvalue. QED.

e.g. $D_h^+ D_h^-$ has $a_0 = \frac{-2}{h^2}$,

$a_{+1} = a_{-1} = \frac{1}{h^2}$, $a_j = 0$ otherwise.

Fast Fourier Transform, FFT.
(Finite Fourier Transform).

$$u \xrightarrow{F} \hat{u}$$

$$\hat{u}_n = c \sum_{j \in B} e^{-2\pi i h n j / L} u_j$$

$$n \in B'$$

Total work $= |B'| \cdot |B| = N^2$.

FFT: compute $\hat{u} = Fu$ in O

$2 \cdot \log_2(N) \cdot N$ multiplies/adds.

Note: modern computer hardware: running time determined by memory / communication time rather than # of ops. The FFT algorithm can be bad for this.

Algorithm for $N = 2^m$: (N = not a power of 2 algorithm is more complicated, depends on the prime factorization of N)

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Take $z_N = e^{-2\pi i h/L} = e^{-2\pi i/N}$.

This has $z_N^N = 1$ and $z_N^k \neq 1$ for $0 < k < N$.

$FFT_N : \mathbb{C}^N \to \mathbb{C}^N$

$u \rightsquigarrow \hat{u} \qquad \hat{u}_n = \sum_{j=0}^{N-1} z_N^{nj} u_j$

$n = 0, \ldots, N-1$.

Suppose $N = 2M$, then $j = 2k$ or $j = 2k+1$

with $k = 0, 1, \ldots, M-1$

$$\hat{u}_n = \sum_{k=0}^{M-1} z_N^{n(2k)} u_{2k} + \sum_{k=0}^{M-1} z_N^{n(2k+1)} u_{2k+1}.$$

Define $v \in \mathbb{C}^M$, $w \in \mathbb{C}^M$ by

$v_k = u_{2k}$, $w_k = u_{2k+1}$. Then

$$\hat{u}_n = \sum_{k=0}^{M-1} (z_N^2)^{nk} v_k + z_N^n \sum_{k=0}^{M-1} (z_N^2)^{nk} w_k$$

$$= \hat{v}_n + z_N^n \hat{w}_n$$

where $\hat{v} = FFT_M v$, $\hat{w} = FFT_M w$.

since $z_N^2 = z_M$.

Note: $\hat{v}_{n+M} = \hat{v}_n$, $\hat{w}_{n+M} = \hat{w}_n$

because $\hat{v}_n$, $\hat{w}_n$ are size M FFT's.

FFT algorithm

input $u \in \mathbb{C}^N$ as an array.

copy $\left.\begin{array}{l} v = \text{evens}(u) \\ w = \text{odds}(u) \end{array}\right\}$ length M arrays.

$\hat{v} = \text{FFT}_M(v)$, $\hat{w} = \text{FFT}_M(w)$

length M arrays.

copy $\hat{u} = \underbrace{\text{double of } \hat{v} + z^n \cdot \text{double of } \hat{w}}$.

arithmetic: N adds + N mults

or $2N$ mults if you

compute z^n on the fly.

if $N = 2^l$, do N adds + $2N$ mults on each

level, total $= \left.\begin{array}{l} l \cdot N \text{ adds} \\ 2l \cdot N \text{ mults} \end{array}\right\} N \log_2(N)$.

For large N on modern hardware, the copies take longer than the arithmetic.

Properties of FFT

- Diagonalize translation invariant operators
- Convolution $\Leftrightarrow$ multiplication.
  - fast convolution, fast polynomial multiplication.
- Plancherel $\|u\|_{\ell^2} = c \cdot \|\hat{u}\|_{\ell^2}$.
- analytical properties ~~smooth u $\Leftrightarrow$ fast conv~~

  smooth $u \Leftrightarrow$ rapid decay of $\hat{u}$.

  We will come back to this after a detour

into ~~the~~ continuous ~~fo~~ Fourier series & transform.

Fourier series. $u(x+L) = u(x)$ has

$$\boxed{u(x) = \sum_{n=-\infty}^{\infty} \hat{u}_n e^{2\pi i n x/L}} \quad (1)$$

The formula for $\hat{u}_n$ - Fourier coefficient -

is "obvious" (multiply by $e^{-2\pi i m x/L}$ and integrate)

$$\hat{u}_n = \frac{1}{L}\int_0^L e^{-2\pi i n x/L} u(x)\,dx \quad (2)$$

The issue is completeness, are the functions

$$v_n(x) = e^{2\pi i n x / L}$$

a basis for periodic fns. This is the same as asking whether the sum (1) with $\hat{u}_n$ defined by (2) converges to $u(x)$.

We will eventually prove this. The proof will rely on the completeness for the DFT, which depends only on $|B| = |B'| = N$.

For applications it is important that the Fourier representation (1) is <u>efficient</u>

$$\left| u(x) - \sum_{n \in B} \hat{u}_n e^{2\pi i n x / L} \right|$$

is very small when ~~the~~ u is smooth.

This is the same as $\hat{u}_n$ being small for $n \notin B$. We see this using integration by parts. Start with

$$\partial_x e^{-2\pi i n x/L} = \frac{-2\pi i n}{L} e^{-2\pi i n x/L}$$

and get

$$e^{-2\pi i n x/L} = \frac{-L}{2\pi i n}\,\partial_x e^{-2\pi i n x/L}$$

Then

$$\hat{u}_n = \frac{1}{L}\int_0^L e^{-2\pi i n x/L}\, u(x)\,dx$$

$$= \frac{1}{L}\cdot \frac{-L}{2\pi i n}\cdot \int_0^L \partial_x e^{-2\pi i n x/L}\, u(x)\,dx$$

$$= \frac{1}{L}\cdot \frac{L}{2\pi i n}\cdot \int^{0} e^{-2\pi i n x/L}\left(\partial_x u(x)\right)dx.$$

Do this p times

$$\hat{u}_n = \frac{1}{L}\cdot\left(\frac{L}{2\pi i n}\right)^p\cdot\int_0^L e^{-2\pi i n x/L}\left(\partial_x^p u(x)\right)dx$$

Define

$$M_p = \max_{0\le x\le L}\left|\partial_x^p u(x)\right|.$$

Then

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$$|\hat{u}_n| \leq \left(\frac{L}{2\pi n}\right)^p \cdot M_p$$

$u(x)$ is analytic at a point x_0 if

The Taylor series converges with a radius of convergence $R(x_0) > 0$. The Taylor series is

$$u(x) = \cancel{u(x)} \sum_{p=0}^{\infty} \partial_x^p u(x_0) \cdot \frac{(x-x_0)^p}{p!}$$

If $|x - x_0| = R < R_0$, then the series converges and the terms are bounded, which mean

$$\left| \frac{\partial_x^p u(x_0) \cdot R^p}{p!} \right| \leq c$$

or

$$(*) \quad \left| \partial_x^p u(x_0) \right| \leq \frac{c \cdot p!}{R^p}$$

Lemma (exercise) If $u(x)$ is analytic at every point x_0, then the inequality $\circledast$ can be taken to be uniform over all x_0:

There is a C ~~for each each p~~ so that

$$\boxed{\left|\partial_x^p u(x)\right| \le C \frac{p!}{R^p}}$$

for all x.

$$\boxed{M_p \le C \cdot \frac{p!}{R^p}}$$

$$|\hat{u}_n| \le C\left(\frac{L}{2\pi n}\right)^p \cdot \frac{p!}{R^p}$$

find the optimal p (a trick I learned from Tom Spencer)

Method 1: Stirling's approximation

$p! = p^p e^{-p} \sqrt{2\pi p}$ + smaller.

differentiate with respect to p &
find the optimal p.

Method 2 (in Feller, probability book, vol I)

Look for p where p and $p+1$
give the same answer:

$$\cancel{\varepsilon^p} \left(\frac{L}{2\pi R n}\right)^{p} \cdot p! = \left(\frac{L}{2\pi R n}\right)^{p+1} \cdot (p+1)!$$

$$p+1 = \frac{2\pi R n}{L}$$

set $p = \dfrac{2\pi R n}{L}$ (& be simple, not greedy)

$$\cancel{|\hat{u}_n| \le c \cdot \left(\frac{L}{2\pi R n}\right)}$$

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Then use Stirling:

$\cancel{p!}$ $\cancel{p!}$

$$p! = \sqrt{2\pi p} \cdot \left(\frac{2\pi Rn}{L}\right)^p \cdot e^{-\frac{2\pi Rn}{L}}$$

$$|\hat{u}_n| \leq \sqrt{2\pi \cdot \frac{2\pi Rn}{L}} \cdot e^{-2\pi Rn/L}$$

~~thus~~

$$|\hat{u}_n| \lessapprox e^{-c \cdot n}$$

where $c \cong \frac{2\pi R}{L}$.