Assignment 3, due March ??

Corrections: (none yet)

1. (*Consistency using symbols*) In this problem we have a discrete function with values U_j, or $U_{n,j}$ if time dependent. We have a function $u(x,t)$ defined for all x and $t \geq 0$. We write the continuous Fourier representation of $u(x,t)$ as

$$u(x,t) = \int_{-\infty}^{\infty} e^{ipx}\, \widehat{u}(p,t)\, dp \; . \tag{1}$$ `cF`

The wave number in this representation is $p = 2\pi/L$, where L is the wave length. The grid function $U_{j,n}$ has Fourier representation

$$U_{j,n} = \int_{-\pi}^{\pi} e^{ikj}\, \widetilde{U}_n(k)\, dk \; . \tag{2}$$ `dFk`

The discrete frequency parameter k can be called the *cell wave number*, because $2\pi/k$ is the number of cells (j values) in one period of e^{ikj}. We want to rescale k to compare U to u when we think $U_{j,n} \approx u(x_j, t_n)$, with $x_j = j\Delta x$ and $t_n = n\Delta t$. The relation is chosen to make the following true: $px_j = kj$, which gives $k = p\Delta x$.

(a) Suppose $u(x,t)$ satisfies the scalar advection equation

$$\partial_t u + s\partial_x u = 0 \; .$$

Show that

$$\widehat{u}(p, t+\Delta t) = m(k, \Delta t)\, \widehat{u}(p,t) \; , \tag{3}$$ `em`

where

$$m(k,\Delta t) = e^{-i\lambda k} \; . \tag{4}$$ `emf`

with $\lambda = s\Delta t/\Delta x$ being the CFL number. Assume the above relation between p and k. This is the symbol of the exact solution to the PDE. It is exactly on the unit circle, $|m(k,\Delta t)| = 1$.

(b) The *Lax Wendroff* finite difference approximation is

$$U_{j,n+1} = U_{j,n} - \frac{s\Delta t}{2\Delta x}\left(U_{j+1,n} - U_{j-1,n}\right) + \frac{s^2\Delta t^2}{2\Delta x^2}\left(U_{j+1,n} - 2U_{j,n} + U_{j-1,n}\right) \; .$$

Define the symbol of this by

$$\widetilde{U}_{n+1}(k) = M(k,\lambda)\, \widetilde{U}_n(k) \; . \tag{5}$$ `M`

Calculate M and show, for fixed λ, that $M(k) = m(k) + O(k^3)$. Show that $|M(k,\lambda)| \leq 1$ for all k if and only if $|\lambda| \leq 1$.

(c) The *trapezoid rule* scheme is the implicit scheme that approximates $\partial_x u$ at time t_n and t_{n+1}:

$$U_{j,n+1} = U_{n,j} - \frac{s\Delta t}{4\Delta x} \left(U_{j+1,n} - U_{j-1,n} + U_{j+1,n+1} - U_{j-1,n+1} \right) \ .$$

Calculate the symbol for this scheme and show that $|M(k,\lambda)| = 1$ for all k. The definition (**??**) still applies, but now you have to solve an algebraic equation to get a formula for M. Do it in the Fourier domain.

(d) A general explicit one step two level scheme has the form

$$U_{j,n+1} = \sum_h a_h(\lambda) U_{j-h,n} \ .$$

If the scheme is explicit with a finite stencil, then there are finitely many non-zero coefficients a_h. The scheme is accurate of order p (this p is not the wave number) if

$$u(x_j, t_{n+1}) = \sum_h a_h(\lambda) u(x_{j-h}, t_n) + \Delta t R_{j,n}$$
$$R_{j,n} = O(\Delta x^p) \ .$$

We always assume $\lambda = s\Delta t/\Delta x$ is fixed. Show that a scheme is accurate of order p if the symbol satisfies

$$m(k) = M(k) + O(k^{p+1}) \ .$$

(e) Use the general method of part (1d) to show that the trapezoid rule is second order accurate.

2. (*Well posedness*) Consider the system of equations

$$\partial_t \rho + a\partial_x u + b\partial_x v = 0$$
$$\partial_t u + c_u^2 \partial_x \rho = 0$$
$$\partial_t v + c_v^2 \partial_x \rho = 0 \ .$$

Show that this is a well posed system of hyperbolic equations with three distinct real propagation modes if and only if the following condition is satisfied: $ac_u^2 + bc_v^2 > 0$.

3. (*A stability computation with applications*) Consider a three point one-sided two level schemes for the first order linear scalar advection equation $\partial_1 u + s\partial_s u = 0$:

$$U_{j,n+1} = a(\lambda)U_{n-2,n} + b(\lambda)U_{j-1.n} + c(\lambda)U_{j,n}$$
$$U_{j,n+1} = a(\lambda)U_{n+2,n} + b(\lambda)U_{j+1.n} + c(\lambda)U_{j,n} \ .$$

Here $\lambda = s\Delta t/\Delta x$ is the CFL number.

(a) Find formulas for the coefficients a, b, and c that make the method second order accurate. This should be possible for both schemes and for any λ.

(b) Show that the geometric CFL condition requires one of these schemes to unstable at any CFL number and the other to be unstable if $\lambda > 2$. The one that is allowed to be stable for $0 < \lambda \leq 2$ is called *upwind differencing*. The other one is called *downwind*. Can you explain this terminology in terms of ordinary wind?

(c) Use von Neumann analysis to find the actual stability limit of the second order three point upwind scheme.

4. (*Multi-Dimensional problems*) Consider the hyperbolic system in two dimensions $\partial_t u + A\partial_x u + B\partial_y u = 0$. Suppose A and B $n \times n$ matrices. Suppose $\Delta x = \Delta y$ and $U_{j,k,n} \approx u(x_j, y_k, t_n)$, with $x_j = j\Delta x$, $y_k = k\Delta y$, and $t_n = n\Delta t$.

   More specifically, suppose $u = (\rho, u_x, u_y)^t$ is the unknown for the 2D linearized compressible gas dynamics equations and A and B are the matrices we gave in class. Define the CFL number to be $\lambda = c\Delta t/\Delta x$. This is a dimensionless measure of the time step.

   (a) Consider the approximations $\partial_t u \leftarrow (U_{j,k,n+1} - U_{j,k,n})/\Delta t$, and $\partial_x u \leftarrow (U_{j+1,k,n} - U_{j-1,k,n})/(2\Delta x)$, and $\partial_y u \leftarrow (U_{j,k+1,n} - U_{j,k-1,n})/(2\Delta x)$. Show that these lead to a scheme that is formally first order accurate but is unstable by von Neumann analysis for any λ.

   (b) Show that the Lax Wendroff idea leads to a second order accurate method with a nine point stencil, which means that $U_{j,k,n+1}$ depends on the nine values $U_{j-1,k-1,n}$, $\ldots$, $U_{j+1,k+1,n}$. Use the geometrical CFL condition to show that the scheme is unstable if $\lambda > 1$. Use the analytical von Neumann stability analysis to show that the scheme is stable for sufficiently small λ. If you have extra time, find the actual stability limit.

SW

5. (*Shallow water equations*) The one dimensional shallow water equations describe the following situation. There is a bottom *bathymetry* given by a function $b(x)$, which is the height of the bottom of a water channel as a function of x (x being the distance along the channel or in the direction the wave is moving). There is water above the bottom, whose height is given by $h(x,t)$. The vertical distance from the surface of the water to the bottom of the channel is $h(x,t) - b(x)$. The water flows mainly in the x direction with a speed that is almost independent of the height, z. The idealized water velocity is $u(x,t)$. The dynamics of $h(x,t)$ and $u(x,t)$ are given by local conservation laws for mass of water and momentum. The model assumes that the water is *incompressible*, and its density of water is a constant, ρ. It turns out that the value of ρ is irrelevant for shallow water dynamics.

(a) Show that the total mass of (two dimensional) water in the channel between $x = x_1$ and $x = x_2$ is (we used $x_1 = a$ and $x_2 = b$ in class, but now b is the bottom, so the notation has to change)

$$\rho \int_{x_1}^{x_2} (h(x,t) - b(x))\, dx \;.$$

Assume that the water flow in the channel at x is horizontal with velocity $u(x,t)$. Show that the mass flux at x is

$$\rho \int_{b(x)}^{h(x,t)} u(x,t)\, dz = \rho u(x,t)\, (h(x,t) - b(x)) \;\; .$$

Use this to derive the conservation of water equation

$$\partial_t h(x,t) + \partial_x \left[u(x,t)\, (h(x,t) - b(x)) \right] = 0 \;. \qquad (6) \quad \boxed{\texttt{cwn}}$$

(b) The $x-$momentum between x_1 and x_2 is

$$\rho \int_{x_1}^{x_2} \int_{b(x)}^{h(x,t)} u(x,t)\, dz dx = \rho \int_{x_1}^{x_2} (h(x,t) - b(x)) u(x,t)\, dx \;.$$

The momentum *advected* across the point x_1 is $\rho(h(x_1,t) - b(x_1))u^2(x_1,t)$. The pressure force at x_1 is

$$p_{tot}(x_1,t) = \int_b^h p(x_1,z,t)\, dz \;.$$

In the shallow water approximation, the pressure at a depth d below the water surface is the weight of the water above, which is $g\rho\, d$, where g is the gravitational constant ($g \approx 10\, m/sec^2$ on earth). Show that the total pressure force at x_1 is

$$\frac{g\rho}{2} (h(x_1,t) - b(x_1))^2 \;.$$

This gives total $x-$momentum flux (or *current*) equal to

$$\rho(h(x_1,t) - b(x_1))u^2(x_1,t) \;+\; \frac{g\rho}{2} (h(x_1,t) - b(x_1))^2 \;.$$

cd

(c) For the next few parts make the *constant depth* assumption that b is a constant independent of x. For simplicity, take this constant to be $b = 0$. Assume there is no momentum transfer between the bottom of the channel and the water in the channel. We will revise this assumption below. Derive the conservation of momentum (more properly, $x-$momentum) equation

$$\partial_t \left(h(x,t) u(x,t) \right) + \partial_x \left(h(x,t) u(x,t)^2 + \frac{g}{2} h(x,t)^2 \right) = 0 \;. \qquad (7) \quad \boxed{\texttt{cmn}}$$

The equation looks nicer with the arguments left out:

$$\partial_t \left(hu \right) + \partial_x \left(hu^2 + \frac{g}{2} h^2 \right) = 0 \;.$$

(d) Suppose $h(x,t) = \overline{h} + \dot{h}(x,t)$, and that $u(x,t) = \dot{u}(x,t)$ with $\dot{h}$ and $\dot{u}$ small. Derive the linearized constant depth shallow water equations, which is a pair of evolution equations for $\dot{h}$ and $\dot{u}$. Show that, in this model, shallow water waves move to the right or to the left with speed

$$s = \sqrt{g\overline{h}}\ .$$

Assume the ocean is 4 *km* deep and that Japan is 8000 *km* from California, and that tsunamis move like shallow water waves in the deep ocean (they do). Show that it takes about 11 hours for the tsunami to cross from Japan to California, which is about how long it takes.

(e) Suppose that the wave is strictly left moving, which means that the solution takes the form $\dot{h}(x,st)$ and $\dot{u}(x-st)$. Find a relation,

$$\alpha\dot{h} + \beta\dot{u} = 0\ ,$$

with fixed constants α and β depending only on $\overline{h}$ and g, that must be satisfied.

vd

6. (*Shallow water equations, variable depth*) This exercise leads to a modification of the shallow water system when we drop the constant depth assumption. The new thing here is that a sloping bottom, which is $\partial_x b(x) \neq 0$, applies a force in the $x-$direction on the water in the channel. The force per unit length is $-p_{bot}(x,t)\partial_x b(x)$, where p_{bot} is the pressure at $z = b(x)$, which is the bottom of the channel. The minus sign is so that the force pushes the water to the right if the channel is sloping downward to the right, which is $\partial_x b(x) < 0$.

(a) Start with the integral $x-$momentum conservation form

$$\frac{d}{dt}\rho\int_{x_1}^{x_2}(h-b)u\,dx = F_{x_2} - F_{x_1} - \int_{x_1}^{x_2} p_{bot}\partial_x b\,dx\ ,$$

where F_{x_1} is the $x-$momentum flux rate (current) at x_1 from before. Assume u and h are smooth and derive a differential equation

$$\partial_t(h-b)u + \partial_x\left[(h-b)u^2 + \frac{g}{2}(h-b)^2\right] = -g(h-b)\partial_x b\ . \qquad (8)$$ vdm

Show that $u = 0$ and $h = \overline{h} = const$ satisfies this equation. This is the sanity check check that water not moving with a flat surface satisfies the equation even in a variable depth channel.

(b) Assume small $\dot{h}$ and $\dot{u}$ and derive the *small disturbance* linearized version of the variable depth shallow water system

$$\partial_t\dot{h} + \partial_x\left((\overline{h} - b(x))\dot{u}\right) = 0 \qquad (9)$$

$$\partial_t\dot{u} + g\partial_x\dot{h} = 0\ . \qquad (10)$$

7. (*Computation*) This is the next step in programming. Download the file `Assignment3.tar`. Put it in a separate directory and unpack it using `tar -xvf Assignment3.tar`. Download the program `ffmpeg` and put it somewhere. Create a directory `WaveMovieFrames` in the directory with the rest of the code. Edit the `Makefile` to changes the lines

   ```
   PYTHON  = /Users/jg/anaconda/bin/python
   FFMPEG  = /Users/jg/bin/ffmpeg
   ```

   so that they represent the correct paths on your system. Then type `make WaveMovie.mpg`. If all goes well, you will get a file `WaveMovie.mpg`, which is an mpeg movie. On a mac, you view the movie typing `open WaveMovie.mpg`.

   Modify the program to solve the linearized shallow water equations with periodic boundary conditions. You have some freedom to try different things, but at a minimum:

   (a) Choose initial data with $h(x, 0)$ looking like the short pulse wave in the demo. Choose initial conditions $u(x, 0)$ that corresponds to a wave moving to the left.

   (b) Make a movie of the wave moving to the left with both the numerical and the exact height (one movie, two curves per frame). Choose $\lambda = .8$ and Δx so that you can see the difference between the computed and exact solutions in the movie. It might help to make the movie long enough so that the wave goes around a few times.

   (c) Put in variable bathymetry. Play with a left moving wave going over a localized disturbance of the boundary and generating a right moving wave.