Monte Carlo Methods, Fall 2007, Courant Institute, NYU
Homework 3, Due November 1

1. The central limit theorem applies to multivariate random variables. If Y_k is a sequence of mean zero i.i.d. two component random variables and $R_n = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} Y_k$, then the distribution of R_n is approximately that of a two component Gaussian for large n. Of course, the covariance matrix of R_n is the same as the covariance matrix of the Y_k. Apply this to the case $Y = (Y_1, Y_2) = (Z, Z^3)$, where $Z \sim \mathcal{N}(0,1)$. Use this to estimate the probability that $\sum_k Z_k \leq \sum_k Z_k^3$ given that $\sum_k Z_k \geq 0$.

2. Consider the SDE $dX = XdW$ with initial condition $X(0) = 1$. The Euler approximation is
$$X_{n+1} = X_n + X_n \Delta W_n \ . \qquad (1)$$
Here, we use the notations $\Delta W_n = W(t_{n+1}) - W(t_n)$, and $X_n = X^h(t_n)$, of course with $t_n = nh$.
   (a) Write a formula that we could use to implement (1) on the computer using $Z_n \sim \mathcal{N}(0,1)$, a sequence of independent standard normals. Hint: Write ΔW_n as a constant multiple of Z_n, figure out the multiple and explain why this gives random variables of the correct distribution.
   (b) Use the result of (a) to write X_n as a product of n factors involving the Z_k.
   (c) Use the formula
$$1 + \epsilon \approx \exp(\epsilon - \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 - \cdots)$$
to write $X_n = \exp(Y_n + U_n + V_n)$, where: Y_n is approximately (or exactly) a Gaussian with a certain $O(1)$ mean and variance, U_n is approximately Gaussian with mean zero and $O(\frac{1}{n})$ variance, and V_n is smaller than U_n (on average). What is the (asymptotic as $n \to \infty$) covariance between Y_n and $n^{-1/2}U_n$?
   (d) Use this to show that the error in the strong sense of the Euler approximation has the exact order of magnitude $1/\sqrt{n}$. Note that to do this you have to reinterpret the Z_k in terms of ΔW_k.

3. Consider the scalar SDE with zero drift $dX(t) = b(X(t))dW(t)$. The Milstein approximation is
$$X_{n+1} = X_n + b(X_n)\Delta W_n + \frac{1}{2}b'(X_n)b(X_n)\big(\Delta W_n^2 - \Delta t\big) \ . \qquad (2)$$
   (a) Take (2) as the definition of $\Phi(x, W[0,h], h)$ in
$$X_{n+1} = X_n + \Phi(X_n, W[t_n, t_{n+1}], h) \ .$$

Let Ψ be defined as in the notes. Show that

$$E\left[\left(\Psi(x,W[0,h],h)\ -\ \Phi(x,W[0,h],h)\right)^2\right]\ \leq\ Ch^3\ .$$

(b) Use this to show that

$$E\Big[\,\big|\,X(t_n)\ -\ X_n\,\big|\,\Big]\ \leq\ C(T)h\ ,$$

if $t_n \leq T$ as $n \to \infty$ as $h \to 0$.

4. Consider the SDE with state dependent noise

$$dX = -\lambda X dt + \left(\sigma_0 + \sigma_2 X^2\right) dW\ , \tag{3}$$

and initial condition $X(0) = 0$. Choose parameters $\lambda = .4$, $\sigma_0 = .2$, and $\sigma_2 = 1$. Write a program to compute approximate sample paths for (3) using the Euler method and Milstein's method. (If the programs are well written, changing from Euler to Milstein should be a very quick minor change.) For each value of h, compute the approximations $X^h(t)$ and $X^{2h}(t)$. For each value of h, generate L sample paths of X^h and X^{2h}, and compute

$$M^h(T) = \max_{0\leq t\leq T}\left|X^h(t) - X^{2h}(t)\right|\ .$$

Note that the maximum is achieved at one of the time step times $t_k = kh$. Take $T = 2$.

(a) Let $f(m,h)$ be the probability density of M^h (at a fixed T). What hypothesis about how f depends on m and h is equivalent to the statement that the random variables M^h scale with h^p but otherwise have the same distribution?

(b) What way of plotting $f(m,h)$ and $f(m,2h)$ (as functions of m) would result in the two curves being identical, if the scaling hypothesis of part (a) is true? Note: It is common practice throughout science and engineering to test scaling hypotheses by attempting to *collapse* data from separate experiments or computations onto a single curve.

(c) Use Monte Carlo data on M^h for various h values to check whether M^h has the h^p scalings suggested by theory ($p = \frac{1}{2}$ for Euler, $p = 1$ for Milstein).