Assignment 6, due April 11

Corrections: (April 8, many correction)

1. **True/False**.
   (a) If A and B are $n \times n$ symmetric matrices, then AB is a symmetric matrix. (Hint: see exercise 2.)
   (b) If (X, Y) is a two dimensional random variable with probability density $p_{XY}(x, y)$, and if $\text{cov}(X, Y) = 0$, then X and Y are independent. (Hint: Suppose (X, Y) is uniformly distributed in the unit disk $x^2 + y^2 \leq 1$. If $X > .9$ then $Y < .9$ (why?).)
   (c) If C is an $n \times n$ matrix with $c_{ij} > 0$ for all i, j, then $w^t C w \geq 0$ for any n component vector w.
   (d) IF C is an $n \times n$ matrix with $c_{ij} = \text{cov}(X_i, X_j)$ for some correlated random variables $X_1, \ldots, X_n$, then $w^t C w \geq 0$ for any n component vector w.

2. Verify that matrix multiplication is associative in this example.
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad , \quad v = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$
   Calculate the vector and matrix
$$b = Av \; , \quad M = vv^t \; .$$
   Show that associativity is true for Avv^t by verifying (using arithmetic) that $bv^t = AM$, which is:
$$Avv^t = [Av]\, v^t = A \left[vv^t \right] = \begin{pmatrix} 56 & 70 \\ 128 & 160 \end{pmatrix} \; .$$

3. Suppose $f(x, y) = x^2 + y^2$ and $g(x, y) = 3x + 4y$.
   (a) Suppose $x_0 = 1$, and $y_0 = 1$. We want to choose Δx and Δy so that $\Delta f = -.1$ and $\Delta g = .2$. Do this using first derivative approximations to estimate Δf (in terms of
$$\nabla f = \text{grad}(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \; .$$
   and ∇g. The answer will not be exact. Using the gradients, you can write Δf and Δg in terms of Δx and Δy. You get two linear equations, which you can solve for Δx and Δy.

(b) Use the method of Lagrange multipliers to find

$$\max\, g(x,y) \quad \text{subject to the constraint} \quad f(x,y) = 1 \ .$$

(Find the optimal point and the optimal value of g.) Make a drawing of the $x-y$ plane that illustrates the "surface" (curve in 2D) $f(x,y) = 1$ and the "surfaces" $g(x,y) = C$ for various C values. Show, in the drawing, that ∇f is proportional to ∇g at the optimal point.

(c) Use the method of Lagrange multipliers to solve the constrained optimization problem

$$\min f(x,y) \quad \text{subject to the constraint} \quad g(x,y) = 5 \ .$$

Show that this is the same point as part (b). In this case, like in mean/variance analysis, maximizing g with a constraint on f is equivalent to minimizing f with a constraint on g.

4. (The Sherman Morrison formula) Suppose A is a symmetric $n \times n$ matrix and $B = A + vv^t$, where v is some n component column vector. Show that

$$B^{-1} = A^{-1} - cA^{-1}vv^tA^{-1} \ .$$

Find a formula for number c. Hint: Find c to make this work:

$$B(A^{-1} - cA^{-1}vv^tA^{-1}) = (A + vv^t)(A^{-1} - cA^{-1}vv^tA^{-1}) = I \ .$$

The calculation uses $A^{-1}A = I$ and $AA^{-1} = I$ and the fact that matrix multiplication is associative (exercise (3)). The expression $v^tA^{-1}v$ is a 1×1 matrix, which means it is an ordinary number. The expression for c might involve dividing by zero, in which case B is not invertible. Otherwise, B is invertible.

5. The *one factor* market model of Markowitz is that the value of asset X_j is

$$X_j = \mu_j + \sigma_j Z_j + \beta_j Z_0 \ , \quad j = 1, \ldots, n \ .$$

The numbers Z_0 and Z_j are independent and random with mean zero and variance $\mathrm{var}(Z_j) = \mathrm{var}(Z_0) = 1$. The Z_j for $j \geq 1$ are *idiosyncratic factors*, which means factors that apply only to X_j. The remaining random variable Z_0 is the *market factor*, which is the same for each X_j. The number β_j is the market loading in X_j. An asset X_j is *beta neutral* if $\beta_j = 0$.

(a) Show that the covariance matrix of X has the form

$$C = D + \beta\beta^t \ ,$$

where D is a diagonal matrix and β consists of the market factor loadings β_j.

(b) Find a formula for C^{-1}. Show that C is invertible. Assume that $\sigma_j > 0$ for $j = 1, \ldots, n$. Hint: use the Sherman Morrison formula. The inverse of a diagonal matrix is diagonal.

Computing exercise.

Write an `R` script that verifies the Sherman Morrison formula problem (4) in some specific cases. Suppose A is an $n \times n$ matrix with entries $a_{ii} = 2$, and $a_{i,i+1} = a_{i+1,i} = 1$ and $a_{ij} = 0$ otherwise. Suppose v is an n component column vector such as $v_i = 1$ for all i or $v_i = 1/i$. Calculate A^{-1} using the `solve()` function in `R`. Use this and the Sherman Morrison formula to calculate $\left(A + vv^t\right)^{-1}$. Suppose M is the answer from the Sherman Morrison formula. Calculate $M(A + vv^t)$ to see whether M is actually the inverse. Try a few sizes n ranging from small to very large. When n is very large, you need a single number to say whether $B = M(A^v v^t)$ is close to the identity matrix. One possibility is

$$R^2 = \sum_{i,j} \left(b_{ij} - \delta_{ij}\right)^2 \ .$$

Here δ_{ij} are the entries in the identity matrix. You should notice that the script takes a while to run when n is large, and that R^2 gets larger (though not actually large). Hand in a printout of your script and some sample output.