Assignment 5, due April 2

Corrections: (none yet)

1. **True/False**.

   (a) If you use the optimal exercise strategy for an American style put, it will generate a positive cash flow if $S_t < K$ (in the money) for some $t < T$ (T = expiration time).

   (b) The expected payout of a European put is smaller in the real world than in with risk neutral probabilities. Use the Black Scholes model (geometric Brownian motion).

2. **Multiple choice** Which of the choices is not true about a skewed probability distribution for S with $\mathrm{E}(S) = 0$? (Think of $S = e^X - C$, where X is Gaussian and C is a constant as an example, but there are many others.)

   (a) It is likely that $\mathrm{E}\left[S^3\right] \neq 0$.

   (b) The median is likely to be different from the mean (see problem 3 below).

   (c) Moments are infinite: $\mathrm{E}[|S|^n] = \infty$.

   (d) S could be Gaussian.

3. If X is a random variable, the *median* is M if $\Pr(X > M) = \Pr(X < M)$. (If there is a probability density, then $\Pr(X = M) = \int_M^M p(x)dx = 0$.) Suppose S_t is a geometric Brownian motion (stock price process) with expected rate of return μ and volatility σ. Then $\mathrm{E}[S_T] = e^{\mu T} S_0$ (we showed in class). Find the median $M_T = \mathrm{median}(S_T)$. Use your answer to explain the statement: "If you choose a stock at random, you are likely to underperform the market." Hint: Suppose X is a random variable with median M_X, and $Y = f(X)$, where $f(x)$ is strictly monotone (increasing or decreasing). Then (explain this) $M_Y = f(M_X)$.

4. Let $V(s, T, \sigma, r, K)$ be the Black Scholes formula for the value of a European put on a stock that pays no dividend if the stock price at time $t = 0$ is s and the expiration time is T. This was given on Assignment 4. This exercise asks you to calculate "the Greeks", derivatives of V with respect to parameters. To do these calculations, you have to use the chain rule and compute quantities like

$$d_1' = \frac{\partial d_1}{\partial s} \ .$$

You also need the formula for the derivative of the cumulative normal distribution function

$$\frac{d}{dz}N(z) = p(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} \; .$$

These calculations can be complicated and error prone. Please show all your work.

(a) Find the Black Scholes formula for

$$\Delta = \frac{\partial V}{\partial s} \; .$$

(b) Find the Black Scholes formula for

$$\Gamma = \frac{\partial^2 V}{\partial s^2} \; .$$

This is called “Gamma” (the Greek letter), or “convexity”.

(c) Find the Black Scholes formula for

$$\Lambda = \frac{\partial V}{\partial \sigma} \; .$$

This is called *Vega*, even though the Greek letter is “Lambda”.

5. Suppose you have a function $f(x)$. Verify the finite difference approximations to the derivatives

$$\begin{aligned} f'(x) &= \frac{f(x+h) - f(x)}{h} + O(h) \\ f'(x) &= \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \\ f''(x) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2) \; . \end{aligned}$$

These mean that when you expand f in a Taylor series about x and do the calculation with enough terms of the Taylor series, the first *error term* is proportional to h (for the first formula) and proportional to h^2 for the other formulas. Write simple `R` scripts using finite difference approximations and the Black Scholes formula that you coded for Assignment 4 to verify that your three “Greek” formulas from problem 4 are correct. You have to use finite differences in the s variable for Γ and Δ, but in the σ variable for Λ.

Computing exercise. This exercise simulates dynamic hedging strategies in various situations. Read the handout `Simulation` (on the `resources page`) before working on this. For this exercise, assume that the underlier (the “stock”) is a geometric Brownian motion. Choose a time step $\Delta t = T/n$ (n equal sized time steps to get from now ($t = 0$) to expiration ($t = T$). The price change at a time step can be $S_{t_{k+1}} = e^{X_k} S_{t_k}$, where the driving "noise" random variables X_k are independent and Gaussian with mean $(\mu - \frac{\sigma^2}{2})\Delta t$ and variance $\sigma^2 \Delta t$. The X_k random variables can be made with the `R` function `randn()`. You have to multiply by $\sigma\sqrt{\Delta t}$ to get the right variance then add $(\mu - \frac{\sigma^2}{2})\Delta t$ to get the right mean. You are welcome to use code, or the code structure, from the code posted with the `Simulation` handout.

Part 1, Dynamic (Delta) hedging. This explores the Delta hedging replication of a vanilla European style put that expires at time T. At time $t_k = k\Delta t$, suppose you have Δ_k shares of the underlier and C_k in cash. Take Δ_k to be the Δ recommended by the Black Scholes formula. At time t_{k+1}, you must *rebalance* because the recommended Δ changes. You find C_{k+1} by first adding risk-free interest to C_k (multiply by $e^{r\Delta t}$) and then using cash to buy (or sell) $\Delta_{k+1} - \Delta_k$ shares of stock at price $S_{t_{k+1}}$. When the simulation reaches time $T = t_n$, the portfolio of cash and stock will not match the cash flow of the put because the hedging did not take place continuously. The *replication error* is a random variable Q. Make some histograms of Q for various values of n. Estimate the mean and standard deviation of Q. Estimate the 1% value at risk for Q (the 1% quantile). Comment on how the replication error depends on the time between rebalancing in this model. Choose an at-the-money-forward strike, $\mu = .1$, $r = .02$, and $\sigma = .3$, and $T = .5$. If you have time, experiment with other parameter values. This should be easy if the code is well automated.

Part 2, transaction cost. In this exercise, suppose there is a *bid/ask* (also called *bid/offer*) spread in the market for the asset S_t, with S_t being the *mid price*. That means that it costs $S_t + \frac{1}{2}\epsilon$ to buy a "share" of the asset and you get $S_t - \frac{1}{2}\epsilon$ if you sell a share. Repeat the discrete time dynamic hedging simulation of part 1, but now with the bid/ask spread. Make histograms of the replication error and see how it depends on n for large n (small time steps). Estimate $\mathrm{E}Q$ for each histogram. What is your conclusion about Delta hedging when there is a bid/ask spread?

Part 3, no-transaction region. It is proposed that you can improve the replication behavior of Delta hedging when there is a bid/ask spread by creating a *no transaction* zone around the Black Scholes recommended Δ. This is a zone with width $h > 0$ so that you don't trade unless $|\Delta_k - \Delta(S_{t_k}, t_k)| > h$. If you do trade, you can trade to the Black Scholes Δ or (this is more complicated but in theory slightly better) to $\Delta(S_{t_k}, t_k) \pm h$. If $|\Delta_k - \Delta(S_{t_k}, t_k)| \leq h$, you are in the *no-transaction* region. Repeat the simulation of part 2 with a dynamic hedging strategy that involves a no-transaction region. Use histograms and estimates of $\mathrm{E}[Q]$ to find a good width h. How much does this improve hedging error for large n?