Mathematics of Finance, Courant Institute, Spring 2019
https://www.math.nyu.edu/faculty/goodman/teaching/MathFin2019/MathFinance.html
Always check the class message board before doing any work on the assignment.

Assignment 1, due February 5

Corrections: (none yet)

1. From *An Introduction to Quantitative Finance*, exercise 1 on page 9.

2. Consider a forward contract on the S&P500 index. The contract date is 6 months from today. The risk free interest rate is 2.5%/year. The spot price of the index today is 2000. Find F, the forward price.

3. (*practice with calculus*) Consider an annuity that pays C per year with m payments per year for T years. Then each payment would have size $\frac{1}{m}C$. Let t_j be the time (date) of payment j. Suppose (for simplicity and without loss of generality) that the present time is $t = 0$. The continuous time interest rate is r. The present value of the payment at time t_j is $e^{-rt_j}\frac{1}{m}C$.

   (a) Write a formula for the total present value of the annuity, of the form

   $$V_m = \sum_{j=1}^{n} *** \;.$$

   The number of payments in total is

   $$n = mT = (\text{payments per year}) \times (\text{number of years}) \;.$$

   (b) Write an explicit formula for V_m. Hint: The sum from part (a) is a geometric series, once you take out the common factor. It helps to write a formula for t_j in terms of j and m. The geometric series formula is $S(z) = z + z^2 + \cdots + z^n = (z - z^{n+1})/(1 - z)$.

   (c) (*review*) Consider an integral

   $$A = \int_0^T f(t)\,dt \;.$$

   Let the interval $[0, T]$ be divided into n equal sized small pieces of length Δt. Define $t_j = j\Delta t$, and let that be the right endpoint of the interval $I_j = [t_{j-1}, t_j]$. The *Riemann sum* approximation to the integral is

   $$A_n = \Delta t \sum_{j=1}^{n} f(t_j) \;.$$

Draw a picture to illustrate A_n as an area that is close to the area that defines A when n is large. This is the usual definition of the integral from calculus (except possibly for using the right endpoint instead of the left endpoint) and you probably saw the picture in a calculus book.

(d) Show that the sum from part (a) is a Riemann sum approximation to an integral:

$$V = C \int_0^T e^{-rt}\, dt \; .$$

Find a formula for Δt in terms of m and find a formula for n in terms of m and T.

(e) Calculate the integral from part (d) to get a simple formula for V. The formula is simpler than the formula for V_m, but it should be clear that V_m converges to V as $m \to \infty$.

(f) (*yield to maturity*). Consider a financial instrument with a price P and a present value V. The present value depends on r, and is the sum of the the present values of all the payments. The instrument could be a zero coupon bond with just one payment, an annuity, a coupon bond (coupon payments and a principal payment), or something more complex. The price is determined by the market (what you can buy or sell the instrument for), but the present value is a theoretical number that is a function of r. The *effective yield to maturity* is the value of r so that $V(r) = P$. You could call it the interest rate *implid* by P.

Consider a ten year annuity ($T = 10$) with $C = \$10{,}000 = \$10^4$. Find the price P that makes the yield to maturity $r = 5\%/year$ if

i. $m = 1$ (annual payments)

ii. $m = 4$ (quarterly payments)

iii. $m = 12$ (monthly payments)

iv. $m = \infty$ (theoretical continuous payment).

Comment on the accuracy of the simple $m = \infty$ approximation when m is not infinite. See Problem (3) before doing this.

(g) From now on, use only the $m = \infty$ formula for the present value $V(r)$. Draw a sketch of the function $V(r)$ for $r \geq 0$. Use the graph to show that there is exactly one r_* (effective yield) so that $V(r_*) = P$ as long as P is not more than the un-discounted value $V_0 = CT$ (the sum of the payments without discounting). The sketch should show what happens as $r \to 0$ and as $r \to \infty$.

(h) The equation $V(r) = P$, which you would solve to get the yield to maturity, is *transcendental* and has no solution formula. We are going to find an approximate formula for r_* under the hypothesis that rT is small. Note that for a ten year annuity with $r = 5\%$, we have

$rT = .5$. It's not clear at the start whether this is small enough for the approximation to be accurate.

Use Taylor expansion (of the function e^x or the function $V(r)$) to find the expansion to second order

$$V(r) \approx \widehat{V}(r) = a_0 + a_1 rT + a_2 r^2 T^2 \ .$$

Replace the exact yield-t-maturity equation with the approximate one $P = \widehat{V}(r)$ and solve. The result should be

$$r = \frac{2}{T}\left(1 - \frac{P}{TC}\right) \ .$$

(i) Use the approximate formula from part (h) to estimate the yield to maturity of a ten year annuity that pays \$10,000/year and costs \$80,000. Note that this is less than the "full value" $V_0 = CT =$ \$100,000.

(j) Use the actual formula $V(r)$ to find the actual present value of the annuity with this r. How close to the target \$80,000?

4. (*introduction to* `R`) Read the posted document `StartingWithR.pdf` from the `Resources` page of the public class web site. Follow instructions to install the `R` app on your computer. Spend a few hours playing with `R` along the lines of the document, but not exactly, to test your understanding. When you are ready, either close and re-open the `R` app or give the command `rm(list=ls()) [enter]`. This command removes everything from the environment so nothing from your earlier `R` session will accidentally effect what you're about to do.

   Work at the command line (without scripting or editors) for this exercise. Define an `R` function `pv(r)` that calculates the formula from Problem (1f) and answers the questions there. It should take m, C, and T from the environment. You will have to use a large m value to simulate $m = \infty$. Experiment to see what m is needed. If you can, make a screen capture of your `R` app with the part relevant to Problem (1f). Print the image and hand it in with the assignment.