Assignment 5. Forward Euler for stochastic differential equations.

Given November 12, due November 26.

Objective: To explore the Forward Euler method for stochastic differential equations.

This assignment involves simulation of the stochastic differential equation

$$dS = rSdS + \sigma SdZ \tag{1}$$

with $S(0) = 1$, $0 \leq t \leq 1$, $r = .06$, and $\sigma = .3$. This stochastic differential equation is not our model of actual movements of the stock price (that has μ instead of r). Rather, it is the "risk neutral martingale measure" that we use in valuation problems. There are several questions below that should be easy to answer if you write a sufficiently modular program.

To solve (1) by the Forward Euler method is to generate (approximate) sample paths starting with $S_0 = S(0) = 1$ with $M = 1/\Delta t$ time steps using

$$S_{k+1} = S_k + rS_k\Delta t + \sigma S_k\sqrt{\Delta t}G_k \ \ , \tag{2}$$

where $S_k \approx S(k\Delta t)$. If $\phi(S)$ is a payout function, then

$$f = E\left[\phi(S)\right] \approx \frac{1}{N}\sum_{j=1}^{N} X_j \ \ ,$$

where the X_j are N values of $\phi(S)$ coming from N independent realizations of S using (2) (a total of MN time steps in all). Remember that whenever we make an MC approximation, we should compute an error bar.

1. Check that your program gets the right answer for a European put struck at $S = 1$ (from the BS formula). Here $\phi(S) = max(1 - S(1), 0)$. If you have enough computer time, check that the bias in the MC estimate is $O(\Delta t)$ by doing a convergence study. For this, you must use so many samples that the error bar is much smaller than Δt.

2. To compute sensitivity with respect to σ, write explicitly the dependence of f on σ:

$$f(\sigma) = E_\sigma\left[\phi(S)\right] \ \ .$$

For the European put above, compute $\partial_\sigma f(\sigma)$ at $\sigma = .3$ in two ways:

a. Finite Differences. Make independent MC estimates of $f(\sigma+\Delta\sigma)$ and $f(\sigma-\Delta\sigma)$ and take

$$\partial_\sigma f(\sigma) \approx \frac{f(\sigma+\Delta\sigma) - f(\sigma-\Delta\sigma)}{2\Delta\sigma} \ \ .$$

Try to choose $\Delta\sigma$ to balance the noise amplification from small $\Delta\sigma$ against the truncation error from large $\Delta\sigma$. This balance depends on the number of samples.

b. Direct Method (also called the "same paths method") Compute $\partial_\sigma S_k$ for each S_k directly from (2).

Which method gives greater accuracy? Compare the answer with the known BS value.

Step 3: Repeat parts (1) and (2) for a "lookback option"

$$\phi(S) = \max_{0 \leq t \leq 1} S(t) \quad .$$

If you have enough computer time, try to see how the bias depends on Δt. It will be different. Here, you will have to use Richardson extrapolation with Δt, $2\Delta t$ and $4\Delta t$ because the exact answer is not known.