Assignment 1.

Given September 17, due October 1.

Objective: To explore the forward Euler finite difference method for the heat equation.

Write a program to apply the forward Euler method to the heat equation on the interval $[0,1]$ with Dirichlet boundary conditions $u(0,t) = 0$, $u(1,t) = 0$. The program should use a grid spacing $\Delta x = 1/M$ for some integer, M, so that $x_0 = 0$ and $x_M = 1$ are physical boundary points. The Dirichlet boundary condition can then be implemented as $U_0^n = 0$ and $U_M^n = 0$ for all n. The unknowns being computed then are U_j^n for $j = 1, 2, ..., M-1$ and for $n = 1, 2, ...$ There should be a subroutine (function, procedure) that does the whole computation up to time $t = 1$ that takes as arguments M, $\lambda = \Delta t/\Delta x^2$, and f (the initial data).

(1) If $f(x) = \sin(\pi x)$, then the exact solution is $u(x,t) = e^{-\pi^2 t/2}\sin(\pi x)$. Show by a computational convergence study for some fixed $\lambda < 1$ that the numerical approximation is second order accurate.

(2) The following initial data have increasing amount of smoothness: $f(x) = [x(1-x)]^{-1/2}$, $f(x) = 1$, $f(x) = [x(1-x)]^{1/2}$, $f(x) = x(1-x)$, and so on. Do a convergence study for each of these to see what the order of accuracy is for more singular initial data and when the eventual second order accuracy sets in. Do not use a formula for the exact solution. Note that the more singular initial data do not satisfy the boundary conditions. This is regarded as a singularity in initial data even (as in the case $f(x) = 1$) when f itself is smooth.