1 Question #1

1.1 Finding the value of $u(t,x)$

First we are interested in finding

$$u(t,x) = \sup_y |y| - \frac{(x-y)^2}{4(T-t)} - (T-t)$$

We can find the sup directly with calculus techniques. Define, for fixed $t$ and $x$, $F(y)$ as follows

$$F(y) = |y| - \frac{(x-y)^2}{4(T-t)} - (T-t)$$

First find the sup on the set $[0, \infty)$

For $y > 0$

$$\frac{dF(y)}{dy} = 1 + \frac{(x-y)}{2(T-t)}$$

The first derivative is zero at the point

$$y^* = x + 2(T-t)$$

The second derivative is negative for all $y$ therefore $F(y)$ will have a local max at $y^*$. However if $x < -2(T-t)$ then $y^* < 0$ contradicting the assumption that $y \geq 0$ Therefore this is not a true critical point in the set $(0, \infty)$ (is not even an element of this set). In this case the sup over $[0, \infty)$ is attained at zero. To see this note that if $x < -2(T-t)$ then

$$\frac{dF(y)}{dy} = 1 + \frac{(x-y)}{2(T-t)} < \frac{(-y)}{2(T-t)} < 0$$
That is, $F$ is decreasing and the sup is reached at zero.
In summary
\[
\sup_{y \geq 0} [F(y)] = \begin{cases} 
F(y^*) & \text{if } x \geq -2(T - t) \\
F(0) & \text{if } x < -2(T - t) 
\end{cases}
\]

**Now find the sup on the set $(\infty, 0]$**

For $y < 0$
\[
\frac{dF(y)}{dy} = -1 + \frac{(x - y)}{2(T - t)}
\]
The first derivative is zero at the point
\[
y^{**} = x - 2(T - t)
\]
Again $y^{**}$ will be a local maximum.
If $x > 2(T - t)$ then $y^{**} > 0$ contradicting the assumption that $y \leq 0$ Therefore this is not a true critical point in the set $(\infty, 0]$ In this case the sup over $(\infty, 0]$ is attained at zero. Since, if $x > 2(T - t)$ then
\[
\frac{dF(y)}{dy} = -1 + \frac{(x - y)}{2(T - t)} > \frac{(-y)}{2(T - t)} > 0
\]
Since we assumed $y < 0$
That is, $F$ is increasing and the sup is reached at zero.
In summary
\[
\sup_{y \leq 0} [F(y)] = \begin{cases} 
F(y^{**}) & \text{if } x \leq 2(T - t) \\
F(0) & \text{if } x > 2(T - t) 
\end{cases}
\]

**Finding the sup on the set $(-\infty, \infty)$**

To find the overall sup we just need to compare between the sups on the two sets $(\infty, 0]$ and $[0, \infty)$
\[
\sup_{y \in (-\infty, \infty)} [F(y)] = \begin{cases} 
\max(F(y^{**}), F(0)) & \text{if } x \in (-\infty, -2(T - t)] \\
\max(F(y^{**}), F(y^*)) & \text{if } x \in [-2(T - t), 2(T - t)] \\
\max(F(0), F(y^*)) & \text{if } x \in [-2(T - t), \infty)
\end{cases}
\]
where
\[
F(y^*) = |y^*| - 2(T - t) = y^* - 2(T - t) = x \text{ in case } y^* \text{ exists.}
\]
\[
F(y^{**}) = |y^{**}| - 2(T - t) = -y^{**} - 2(T - t) = -x \text{ in case } y^{**} \text{ exists, and finally}
\]
\[
F(0) = -\frac{x^2}{4(T - t)} - (T - t)
\]
After substitution of these values we find
\[
\sup_{y \in (-\infty, \infty)} [F(y)] = \begin{cases} 
x & \text{if } x \in (-\infty, -2(T - t)] \\
-x & \text{if } x \in [-2(T - t), 0] \\
x & \text{if } x \in [0, 2(T - t)] \\
x & \text{if } x \in [-2(T - t), \infty)
\end{cases}
\]
Therefore \( u(t, x) = |x| \) which is independent of \( t \)

Now if we slightly change the problem to find

\[
u(t, x) = \sup_y [-|y| - \frac{(x - y)^2}{4(T - t)} - (T - t)]
\]

we can proceed in a similar way, but the result will be quite different, in particular \( u \) will depend also in \( t \). Proceeding as we did before:

First find the sup on the set \([0, \infty)\)

For \( y > 0 \)

\[
\frac{dF(y)}{dy} = -1 + \frac{(x - y)}{2(T - t)}
\]

The first derivative is zero at the point

\( y^* = x - 2(T - t) \)

In this case if \( x < 2(T - t) \) then \( y^* < 0 \) and there is no a critical point in \((0, \infty)\) and the sup on this set is attained at zero \((F \) would be is decreasing).

Now the sup on the set \((-\infty, 0]\)

For \( y < 0 \)

\[
\frac{dF(y)}{dy} = 1 + \frac{(x - y)}{2(T - t)}
\]

The first derivative is zero at the point

\( y^{**} = x + 2(T - t) \)

In this case if \( x > -2(T - t) \) then \( y^{**} > 0 \) and there is no a critical point in \((-\infty, 0]\) and the sup on this set is attained at zero \((F \) would be increasing).

Finding the sup on the set \((-\infty, \infty)\)

\[
u(t, x) = \sup_{y \in (-\infty, \infty)} F(y) = \begin{cases} \max(F(y^{**}), F(0)) & \text{if} \ x \in (-\infty, -2(T - t)] \\ F(0) & \text{if} \ x \in [-2(T - t), 2(T - t)] \\ \max(F(0), F(y^{**})) & \text{if} \ x \in [2(T - t), \infty) \end{cases}
\]

where

\[
F(y^*) = -|y^*| - 2(T - t) = y^* - 2(T - t) = -x \quad \text{in case } y^* \text{ exists},
\]
\[
F(y^{**}) = -|y^{**}| - 2(T - t) = -y^{**} - 2(T - t) = x \quad \text{in case } y^{**} \text{ exists}, \quad \text{and finally}
\]
\[
F(0) = -\frac{1}{4(T - t)} - (T - t)
\]

Whenever \( y^* \) or \( y^{**} \) exists we have \( F(y^*) > F(0) \) and \( F(y^{**}) > F(0) \) because,
\[
\begin{align*}
\left( \frac{x}{\sqrt{T-t}} + \sqrt{T-t} \right)^2 &> 0 \\
\left( \frac{x}{\sqrt{T-t}} - \sqrt{T-t} \right)^2 &> 0
\end{align*}
\]

Therefore

\[
\begin{align*}
F(y^{**}) &= x - \frac{x^2}{4(T-t)} - (T-t) = F(0) \\
F(y^*) &= -x - \frac{x^2}{4(T-t)} - (T-t) = F(0)
\end{align*}
\]

Leading to

\[
u(t, x) = \sup_{y \in (-\infty, \infty)} \left[ F(y) \right] = \begin{cases} 
\frac{x^2}{4(T-t)} - (T-t) & \text{if } x \in (-\infty, -2(T-t)] \\
-x & \text{if } x \in [-2(T-t), 2(T-t)] \\
-\frac{x^2}{4(T-t)} - (T-t) & \text{if } x \in [-2(T-t), \infty)
\end{cases}
\]

1.2 Verification of the Viscosity Solution Condition

The Hamilton Jacobi equation associated to the variational problem:

\[
u(t, x) = \sup_y \left\{ \frac{|x|^2}{4(T-t)} - (T-t) \right\}
\]

is given by

\[
\begin{cases} 
u_t + g(\nu_x) = 0 \\
\nu_T = |x|
\end{cases}
\]

where \(g(x) = \sup_y [xy - \frac{y^2 + 4}{4}] = x^2 - 1\)

Trivially \(u(t, x) = |x|\) satisfies the Hamilton Jacobi PDE for \(x \neq 0\)

Is this a viscosity solution?

In principle we need to verify that \(u(t, x) = |x|\) satisfies for all \(t \in [0, T]\) and all \(x \in (-\infty, \infty)\) the definition of viscosity solution.

However at any point \((t_0, x_0)\) of a solution of the PDE where the first derivatives exist, the solution will satisfy the viscosity solution conditions at that point.

To see this note that if \(v\) is a smooth function lying below (above) \(u\) in a neighborhood of \((t_0, x_0)\) and such that \(u = v\) at that point.

That is \((u - v)\) has a local min (max) at that point. Then

\[
\frac{d(u - v)}{dt} = 0, \quad \frac{d(u - v)}{dx} = 0
\]

That is,

\[
\frac{du}{dt} = \frac{dv}{dt}, \quad \frac{du}{dx} = \frac{dv}{dx}
\]
at \((t_0, x_0)\) Therefore \(v\) solves the Hamilton Jacobi PDE at that point and trivially \(\text{since equality holds} \ v_t + v_x^2 - 1 \leq 0\) \((t_0, x_0)\) and also \(v_t + v_x^2 - 1 \geq 0\) \((t_0, x_0)\) satisfying the viscosity solution condition.

Therefore we need to check for \(u = |x|\) the definition of viscosity solution holds only where any of the derivatives is not well defined, i.e. at points of the form \((t, 0)\).

First, at a fixed point \((t_0, 0)\), for a smooth test function time independent \(v\) such that \((u-v)\) has a local min. In this case since \(v\) has to lie below \(u\) the derivative of \(v\) with respect to \(x\) at the origin has to be bounded. In particular

\[
\frac{dv}{dx} \leq 1
\]

To see this note that if this condition does not hold, then a neighborhood of the origin will contain points such that \(v\) would lie above \(u\) since the derivative of \(v\) is larger (in absolute value) than the derivative of \(u\).

**NOTE:** This is just an heuristic argument for what was rigorously proven in class and posted in the course notes.

Similarly it can be proven, that in case \(v\) is not time independent, The condition of "lying below \(u\)" forces the time derivative of \(v\) to vanish at the point of interest \((t_0, 0)\)

Then

\[
v_t + v_x^2 - 1 = v_x^2 - 1
\]

\[
v_x^2 - 1 \leq 0
\]

Therefore satisfying the viscosity solution condition for smooth functions lying below.

Finally, note there are no smooth function \(v\) that lie above \(u\) and touch \(u\) at \((t_0, 0)\), the point of discontinuity of \(u\) at time \(t_0\).

**Therefore the viscosity solution is satisfied in any case.**

Is \(u(t, x) = -|x|\) a viscosity solution when \(\phi(x) = -|x|\) ?

The set of points where we have to check the solution satisfies the viscosity solution conditions is for points of the form \((t, 0)\).

In this case we can only test with smooth functions \(v\) that lie above the candidate for viscosity solution \(u\), in this case \((u - v)\) has a local max.
By similar arguments as the previous case since \( v \) has to lie above \( u \) the derivative of \( v \) with respect to \( x \) at \( t_0 \) has the same bounds.

\[
\frac{dv}{dx} \leq 1
\]

To see this note that if this condition does not hold, then a neighborhood of the origin will contain points such that \( v \) would lie below \( u \) since the derivative of \( v \) is larger (in absolute value) than the derivative of \( u \). Again the condition of "lying above u" forces the time derivative of \( v \) to vanish at the point of interest \((t_0, 0)\)

Then

\[
v_t + v_x^2 - 1 = v_x^2 - 1
\]

\[
v_x^2 - 1 \leq 0
\]

However for the solution to be a viscosity solution we require precisely the other inequality namely \( v_t + v_x^2 - 1 \geq 0 \) since now \( u - v \) has a local max. Showing this is not a viscosity solution.

2 Question #2

For simplicity assume \( g(x) \) is strictly convex. What we know about the problem is:

- \( a < b \)
- \( g(a) = g(b) = 0 \)
- \( g(x) < 0 \) for \( x \in (a, b) \) by strict convexity
- \( f_1(x), f_2(x) \) are piecewise linear.
- \( \frac{df_i}{dx} \) is piecewise constant. For instance, \( \frac{df_i}{dx} = a \) for \( x < 0 \) and \( \frac{df_i}{dx} = b \) for \( x > 0 \). Depending on how you define the functions the right (left) derivative will be equal to either \( a \) or \( b \) in any case the first derivative will not be well defined at the origin.

Trivially the functions \( f_i(x) = w(x, t) \) solve the given HJB PDE for points \((x, t)\) such that \( x \neq 0 \)

\[
\begin{cases}
    w_t + g(w_x) = 0 \\
    u|_{t} = f_i(x)
\end{cases}
\]

Since \( \frac{df_i}{dx} = 0 \) and \( \frac{df_i}{dx} \) is equal either to \( a \) or \( b \). In any case \( g(\frac{df_i}{dx}) = 0 \) is equal to either \( g(a) \) or \( g(b) \), in both cases vanishes.
2.1 testing the viscosity solution condition for $f_1$

Again we only have to check for points of the form $(t, 0)$.

For $f_1$ the first derivative jumps from $a$ to $b$ when crossing the origin. Therefore there are no smooth functions $v$ that stay above $f_1$ and touch $f_1$ at the point of interest namely at $(t_0, 0)$ Now, testing the viscosity solution condition at $t_0$. For a smooth functions $v$ lying below $f_1$ such that $f_1(0) = v(t_0, 0)$, therefore $(f_1 - v)$ has a local min the following bounds on the first derivative must be satisfied. The arguments are similar to what we did for problem1.

$$\frac{dv}{dt} \in [a, b]$$

$$\frac{dv}{dx} = 0$$

Therefore

$$\frac{df_1}{dt} + g(\frac{df_1}{dx}) = g(\frac{df_1}{dx}) \leq 0$$

The last inequality holds because $g(x) < 0$ for $x \in (a, b)$ and $\frac{dv}{dx} \in [a, b]$

Therefore the viscosity solution condition is satisfied in this case.

2.2 testing the viscosity solution condition for $f_2$

We know $f_2$ can not be viscosity solution since viscosity solutions are unique. To verify this fact we can proceed as we did for the function $f_1$.

There are no smooth functions $v$ that stay below $f_2$ and touch $f_2$ at the point of interest. For a smooth $v$ lying above $f_2$ such that $f_2(0) = v(t_0, 0)$, therefore $(f_2 - v)$ has a local max the same bounds hold.

$$\frac{dv}{dx} \in [a, b]$$

$$\frac{dv}{dt} = 0$$

Therefore

$$\frac{df_1}{dt} + g(\frac{df_1}{dx}) = g(\frac{df_1}{dx}) \leq 0$$

However the viscosity solution condition requires precisely the other inequality namely $\frac{df}{dx} + g(\frac{df}{dx}) \leq 0$ since now $f_2 - v$ has a local max. Therefore $f_2$ is not a viscosity solution.