Constructing an elementary measure on a space of projections

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Abstract

Bypassing much theory from integral geometry, we construct an elementary measure on a space whose elements can represent rank $k$ orthogonal projections in $\mathbb{R}^N$. By replacing the Grassmannian $G_{N,k}$ with a simple product space $\bigotimes_{j=1}^k S^{N-1}$ we are able to reproduce certain important features of the nontrivial measure on $G_{N,k}$ invariant under the action of the orthogonal group. As a motivating example we show that our construction enables the proof of a recent embedding theorem due to Foias & Olson to be completed using only standard methods of analysis.

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A simple measure for projections

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1 Introduction

In [3] Foias and Olson proved the following result:

**Theorem 1.1** Let $H$ be a real Hilbert space and $X \subset H$ be such that its fractal (upper box-counting) dimension is less than $m/2$, $d_f(X) < m/2$. Then if $P_0$ is an orthogonal projection of rank $m$ and $\delta > 0$, there is an orthogonal projection $P$ such that $\|P - P_0\| < \delta$ and $P|_X$ has Hölder inverse.

This theorem is the best result in a series of papers, and in particular it generalises the results of Ben-Artzi et al. [1] and Eden et al. [2] which apply only when $X$ is a subset of a finite-dimensional space\(^1\).

One of the main applications of this theorem is in the theory of finite-dimensional global attractors for dissipative partial differential equations (the monographs by Hale [4] and Temam [8] contain numerous examples of such attractors). Since the attractors have finite dimension, it is natural to try to reproduce the dynamics on these attractors in some finite-dimensional system. A first step in any such construction (see the monograph by Eden et al. [2], chapter 10, and Robinson [6], chapter 16) is the embedding of the attractor into a finite-dimensional space.

While the proof in Eden et al. [2] uses only standard methods, the paper Ben-Artzi et al. [1] uses ideas of integral geometry as found in Santalo [7]. These ideas are used again and more extensively in the proof of Foias & Olson [3]. Let us sketch the background which is necessary to derive the result needed in these two papers.

When dealing with orthogonal projections of rank $m$ in $\mathbb{R}^N$, it is natural to identify these projections either with the space of their ranges (as in [1])

\[^1\]There is now a more general result available due to Hunt & Kaloshin [5] that also gives explicit estimates on the Hölder exponent of $P_X^{-1}$.\]
or (as in [3]) of their kernels; that is we identify the rank $m$-projections with the space of $m$, or of $k := N - m$, dimensional linear subspaces of $\mathbb{R}^N$. Either way we are lead to consider linear subspaces of a given dimension (say $k$) in $\mathbb{R}^N$; $G_{N,k}$, the Grassmannian manifold, denotes the space of all such $k$-planes.

We can identify a $k$-plane in $G_{N,k}$ by the (equivalence) class of those elements in the orthogonal group $O(N)$ leaving this $k$-plane invariant,

$$G_{N,k} \leftrightarrow O(N)/(O(k) \times O(N - k)).$$

Since the orthogonal group $O(N)$ is a Lie Group with dimension $N(N - 1)/2$ the space $O(k) \times O(N - k)$ can be identified with a closed subgroup. It is well known in Lie Group Theory that factorising a Lie Group by a closed subgroup still gives a manifold, called a *homogeneous manifold* or a *homogeneous space*. Therefore $G_{N,k}$ appears as a (homogeneous) manifold, generally called the *Grassmann manifold* or *Grassmannian*. This construction can be found, for instance, in Warner [9]. Note that $O(N)$ still acts on $G_{N,k}$ in a natural way (rotating $k$-planes), and is said to be a *Lie Transformation Group* of $G_{N,k}$ (see [7, Chapter 10.5]).

Within this general framework a nontrivial measure $\mu$ on $G_{N,k}$ that is invariant under $O(N)$ can be constructed using standard methods, as discussed in Chapter 10 of [7]. The measure of the whole space $G_{N,k}$ is then computed in Chapter 12 (equation (12.35)), and is

$$\mu(G_{N,k}) = \frac{O_{N-1} \cdots O_{N-k}}{O_{k-1} \cdots O_0}. \quad (1)$$

The application of this measure construction in the papers discussed above is restricted to the following formula, essentially found as equation (17.52) in [7] and here given in the notation of Foias & Olson [3]. Before writing it down we note that deriving this formula (even given the construction of
the measure above) requires knowledge of non-Euclidean integral geometry and projective spaces (see [7, Chapter 17]).

The formula gives the measure of those planes which intersect a given spherical subset of \( \mathbb{R}^N \). If \( B_\rho(a) \) is a ball centred at \( a \) and of radius \( \rho \), with \( a \neq 0 \) and \( \rho < |a| \), and \( S(B) \) denotes the set of all \( k \)-planes having nonempty intersection with \( B \),

\[
S(B) := \{ x \in G_{N,k} : x \cap B \neq \emptyset \},
\]

then

\[
\mu(S(B_\rho(a))) = \frac{O_{N-2} \cdots O_{N-k-1}}{O_{k-2} \cdots O_0} \int_0^{\arcsin(\rho/|a|)} \cos^{k-1}(s) \sin^{N-k-1}(s) \, ds,
\]

where \( O_i \) denotes the surface area of the \( i \)-dimensional unit sphere.

In fact what is really used in [3] is (4) below, a corollary of (2), proved by some simple estimates and by using \( O_{N-1} = 2^{N/2}/\Gamma(N/2) \):

\[
\mu(S(B_\rho(a))) \leq \frac{1}{N-k} \cdot \frac{O_{N-2} \cdots O_{N-k-1}}{O_{k-2} \cdots O_0} \arcsin^{N-k}(\rho/|a|)
\leq \frac{\mu(G_{N,k}) O_{k-1} O_{N-k-1}}{N-k} \left( \frac{\pi \rho}{2|a|} \right)^{N-k}
= \frac{\mu(G_{N,k}) O_{N-k-1} \Gamma(N/2)}{N-k} \left( \frac{\rho}{2|a|} \right)^{N-k} \pi^{(N-k)/2}
\leq \frac{\mu(G_{N,k}) O_{N-k-1}}{N-k} \left( \frac{N\pi}{2} \right)^{(N-k+1)/2} \left( \frac{\rho}{2|a|} \right)^{N-k} \cdot \frac{\Gamma(k/2)}{\Gamma(N/2)}
\]

Readers not familiar with integral geometry, and perhaps those who are, will presumably agree that the above route is a long way to obtain (4). It is therefore interesting that there is an elementary construction giving essentially the same result, so that the entire proof of Theorem 1.1 can be based on standard methods from analysis.
We shall prove essentially the same result as (3), from which (4) will be an immediate consequence. The main simplification comes from working in a different measure space,

\[(S_{N,k}, \nu) \neq (G_{N,k}, \mu),\]
as discussed in detail in the next section.

Let us write (4) again as

\[\mu(S(B_\rho(a))) \leq K(N, k) \left( \frac{\rho}{|a|} \right)^{N-k}\]

with

\[K(N, k) := \mu(G_{N,k}) \frac{O_{N-k-1}}{N-k} \left( \frac{N\pi}{2} \right)^{(N-k+1)/2} \left( \frac{1}{2} \right)^{N-k} ; \]

we emphasise the dependence of the constant on \(N\) and \(k\). It is worth noting that the proof in Ben-Artzi et al. [1] uses this estimate but does not need an explicit form for \(K(N, k)\) as it treats only the finite dimensional case \((N\ \text{fixed})\). The Foias & Olson proof [3], however, keeps careful track of this dependence in order to cope with the infinite-dimensional case.

\section{The measure construction}

We are interested in a measure for \(k\)-dimensional (linear) subspaces in \(\mathbb{R}^N\), simply called \(k\)-planes. Take any \(k\) set of vectors spanning such a plane. Of course we can normalise these vectors, getting \(k\) points on the unit sphere in \(\mathbb{R}^N\). Therefore we take (as our measure space) the product of \(k\) \(N-1\) dimensional unit spheres \(S_{N-1} \subset \mathbb{R}^N\),

\[S_{N,k} := S^{N-1} \times \ldots \times S^{N-1} \quad (k \text{ times}),\]
the measure $\nu = \nu_k = \nu_{N-1,k}$ being the product measure of the uniform surface measures, each of which can be written as $\nu_1$ (or $\nu$ as well if no confusion is possible).

Returning to our representation of $k$-planes in this space, we have already noted that a $k$-plane can be represented by $k$ points on the unit sphere, the corresponding vectors being linearly independent. In fact, almost all $k$-tuples on the unit sphere span a $k$-plane:

**Lemma 2.1** Almost all (with respect to $\nu_{N-1,k}$) $k$-tuples of unit vectors in $\mathbb{R}^N$ span a $k$-plane. More exactly, let

$$D := \{(x_1, \ldots, x_k) \in S^{N-1} \times \ldots \times S^{N-1} : \text{the } x_i \text{ are linearly dependent}\};$$

then $\nu(D) = 0$.

We omit the proof, which is a trivial exercise in measure theory. However, this simple result is the main reason why $S_{N,k}$ is a suitable space in which to investigate the measure for $k$-planes.

## 3 The main result

We will prove now the following estimate corresponding to (5) in the first section.

**Theorem 3.1** Let $B_\rho(a)$ a ball with radius $\rho < |a|$ and centre $a \neq 0 \in \mathbb{R}^N$.

Define the set

$$S(B_\rho(a)) := \{(x_1, \ldots, x_k) \in S^{N-1} \times \ldots \times S^{N-1} : [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset\},$$

where $0 < k < N$. Then there is a constant $K = K(N,k)$ such that:

$$\nu(S(B_\rho(a))) \leq K(\rho/|a|)^{N-k} \quad (6)$$

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Restricting \( \rho \) to \((0, |a|/\pi)\), a possible choice for the constant is

\[
K(N, k) = 2 \cdot \nu(S_{N,k}) \frac{O_{N-k-1}}{N - k} \left( \frac{N\pi}{2} \right)^{(N-k+1)/2} \left( \frac{1}{2} \right)^{N-k} .
\]  

(7)

Before the proof, it will be useful to recall some properties of \( N \) dimensional polar coordinates. In addition to the radius \( r \) we have the angles \( \varphi \in [0, 2\pi) \) and \( \theta_1, \ldots, \theta_{N-2} \in [0, \pi] \), so that in cartesian coordinates a point of \( S^{N-1} \) can be written as

\[
\begin{pmatrix}
\cos \varphi \sin \theta_1 \\
\sin \varphi \sin \theta_1 \\
\cos \theta_1 \sin \theta_2 \\
\vdots \\
\cos \theta_{N-2} \\
\end{pmatrix}
\]

Note that the \( i \)th component is the scalar product with the \( i \)th unit vector \( e_i \) and this is in general not the cosine of one of our angles; this is true only for the last component, \( \cos \theta_{N-2} \). Strictly one should denote the surface element as \( d\nu = d\nu_{N-1,k} \) but it will often be enough to emphasise the \( N \)-dependence, so we adopt the following notation: the surface element of \( S^{N-1} \) is

\[
dS^{N-1} = \sin^{N-2} \theta_{N-2} \sin \theta_1 \sin \theta_{N-2} \ldots d\theta_{N-2} \ldots d\theta_1 d\varphi.
\]

It will be useful to note the recurrence relation,

\[
dS^{N-1} = \sin^{N-2} \theta_{N-2} d\theta_{N-2} dS^{N-2} = \sin^{N-2} \vartheta d\vartheta dS^{N-2} \quad \text{with } N \geq 3.
\]  

(8)

As indicated in the above formula we will write \( \vartheta \) instead of \( \theta_{N-2} \) because it will turn out to be the only angle of interest. \( \vartheta_i \) will denote this angle for the point \( x_i \) (this is a point on the \( i \)th sphere in our product space) defined
in a cartesian system where we will take $a$ or $x_1$ for $e_n$ (or, geometrically, as the North Pole). Finally we note that by integrating (8) we obtain
\[ \int_{\theta \in [0,\pi]} \sin^{N-2} \theta \, d\theta = O_{N-1}/O_{N-2}, \] (9)
where $O_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of the $N$-dimensional unit sphere.

Proof (Theorem 3.1). Note that the condition $[x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset$ with $a \neq 0$ is equivalent to the condition $[x_1, \ldots, x_k] \cap B_{\rho/|a|}(a/|a|) \neq \emptyset$ so it is sufficient to prove the case when $|a| = 1$, i.e. for $a \in S^{N-1}$. We first treat the case $k = 1$:

**Lemma 3.2** Let $N \geq 2$, $|a| = 1$ and $0 < \rho < 1$. Then we have
\[ \nu(x_1 \in S^{N-1} : [x_1] \cap B_\rho(a) \neq \emptyset) = \arcsin \rho \int_{\vartheta_1=0}^{\arcsin \rho} \sin^{N-2} \vartheta_1 \, d\vartheta_1 \] (10)
\[ \leq \frac{2O_{N-2}(\pi/2)^{N-1}}{N-1} \rho^{N-1} =: K(N,1)\rho^{N-1}. \] (11)

**Proof (Lemma 3.2).** If $\vartheta_1$ denotes the angle between $a$ and $x_1$, then the following equivalences hold:
\[ [x_1] \cap B_\rho(a) \neq \emptyset \iff \text{projection of } a \text{ on } [x_1] \text{ has length } \geq (1 - \rho^2)^{1/2} \]
\[ \iff |(x_1,a)x_1|^2 \geq 1 - \rho^2 \]
\[ \iff |(x_1,a)|^2 = \cos^2 \vartheta_1 = 1 - \sin^2 \vartheta_1 \geq 1 - \rho^2 \]
\[ \iff \sin \vartheta_1 \leq \rho \]
This gives
\[ \nu(x_1 \in S^{N-1} : [x_1] \cap B_\rho(a) \neq \emptyset) = \nu(x_1 \in S^{N-1} : \sin \vartheta_1 \leq \rho) \]
\[
\begin{align*}
&= \int_{S^{N-1}} dS^{N-1} \\
&= \int_{S^{N-2}} dS^{N-2} \sin^{N-2} \vartheta_1 \vartheta_1 \vartheta_1 \sin^{N-2} \vartheta_1 \vartheta_1 \\
&= 2 O_{N-2} \int_{\sin \vartheta_1 \leq \rho} \sin^{N-2} \vartheta_1 \vartheta_1 \\
&\leq 2 O_{N-2} \int_{\vartheta_1 = 0} \vartheta_1^{N-2} \vartheta_1 \\
&\leq \frac{2 O_{N-2}}{N-1} \arcsin \rho
\end{align*}
\]

Note that the expression (12) is exactly 2 times the value of the Grassmannian formula (2), applied to \( k = 1 \). This is not surprising, since in the case \( k = 1 \) each choice of one point at the unit sphere leads to a one-dimensional subspace. On the other hand, given a one-dimensional subspace there are exactly 2 possible choices, corresponding to the two intersections of a line through the origin with the unit sphere.

Proof (Theorem 3.1, continued). By the preceding lemma the case \( k = 1 \) is already proved. For the rest we shall use induction on \( N_0 \), where \( 0 < k < N \leq N_0 \). The first possible value for \( N_0 \) is 2, so \( k \) has to be 1 and we can start the induction.

Let us turn to the induction step \( N_0 - 1 \mapsto N_0 \). For \( N < N_0 \) the assumption applies directly, so we only have to consider \( 0 < k < N = N_0 \), and we can even assume \( 1 < k < N = N_0 \). Then

\[
\begin{align*}
&\nu_k(x_1, \ldots, x_k) \in S^{N-1} : \ [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset \\
&= (O_{N-1})^{k-1} \nu_1(x_1) \in S^{N-1} : \ [x_1] \cap B_\rho(a) \neq \emptyset
\end{align*}
\]
\[ +\nu_k(x_1, \ldots, x_k \in S^{N-1} : [x_1] \cap B_\rho(a) = \emptyset \text{ and } [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset) \leq (O_{N-1})^{k-1} K(N, 1)\rho^{N-1} \]

\[ +\nu_k(x_1, \ldots, x_k \in S^{N-1} : [x_1] \cap B_\rho(a) = \emptyset \text{ and } [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset) \]

Using the induction hypothesis we can show

\[ \nu_k(x_1, \ldots, x_k \in S^{N-1} : [x_1] \cap B_\rho(a) = \emptyset \text{ and } [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset) \leq c \cdot K(N - 1, k - 1)\rho^{N-k}, \]

with \( c = c(N, k) = (O_{N-1}/O_{N-2})^{k-1} \cdot O_{N-2}O_{k-1}/O_{k-2}. \) (15)

Indeed, take \( x_1, \ldots, x_k \in S^{N-1} \) such that

\[ [x_1] \cap B_\rho(a) = \emptyset \quad \text{and} \quad [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset. \]

Note that there is at least one vector linearly independent of \( x_1, \) but in fact, due to proposition 2.1, we can even take all the vectors linearly independent. Let \( P \) denote the orthogonal projection in direction \( x_1; \) then

\[ P([x_1, \ldots, x_k] \cap B_\rho(a)) \]

is nonempty (otherwise \( [x_1, \ldots, x_k] \cap B_\rho(a) \subset \ker P = [x_1] \) which is not possible). As the inclusion \( f(A \cap B) \subset f(A) \cap f(B) \) holds for any function \( f, \) we find that \( P([x_1, \ldots, x_k]) \cap PB_\rho(a) \neq \emptyset; \) by abuse of notation we will find it convenient to write \( B_\rho(Pa) \) instead of \( PB_\rho(a). \) Let \( \vartheta_1 = \angle(x_1, a) \) and \( \vartheta_j = \angle(x_1, x_j) \) \((j = 2, \ldots, k)\) and note that \( \sin \vartheta_i \neq 0; \) in fact we have \( \sin \vartheta_i > 0, \) for \( i = 1, \ldots, k. \) Now define

\[ \hat{x}_j := Px_j/\sin \vartheta_j \quad \text{for } j = 2, \ldots, k \]

and \( \hat{a} := Pa/\sin \vartheta_1. \)
As $x = Px + (x_1, x)x_1$ for any $x$ we find that $|Px| = \sin \angle(x_1, x)$ for any $x$ having unit length. Therefore we have $\hat{x}_i, \hat{a} \in S^{N-2}$. That is, we have scaled the projected vectors back to unit length. Note the following:

$$[x_1] \cap B_\rho(a) = \emptyset \quad \text{and} \quad [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset$$

$$\Rightarrow \quad [Px_2, \ldots, Px_k] \cap B_\rho(Pa) \neq \emptyset$$

$$\Leftrightarrow \quad [\hat{x}_2, \ldots, \hat{x}_k] \cap B_{\rho/\sin \vartheta_1}(\hat{a}) \neq \emptyset.$$  

Since $[x_1] \cap B_\rho(a) = \emptyset$ it follows that we must have $\rho/\sin \vartheta_1 < |\hat{a}|$ (this observation will prove useful below).

Note also the correspondence between $x_j$ and $\hat{x}_j \ (j = 2, \ldots, k)$: $x_j \in [x_1, \hat{x}_j]$ and therefore

$$x_j \in (S^{N-1}, dS^{N-1}) \quad \longleftrightarrow \quad (\hat{x}_j, \vartheta_j) \in (S^{N-2} \times [0, \pi], \sin^{N-2} \vartheta_j d\vartheta_j dS^{N-2}),$$

using the notation $(\text{space, measure})$. Now we can write

$$\nu(x_1, \ldots, x_k \in S^{N-1} : [x_1] \cap B_\rho(a) = \emptyset, [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset)$$

$$\leq \nu(x_1, \ldots, x_k \in S^{N-1} : [x_1] \cap B_\rho(a) = \emptyset, [\hat{x}_2, \ldots, \hat{x}_k] \cap B_{\rho/\sin \vartheta_1}(\hat{a}) \neq \emptyset)$$

$$= \int \text{above conditions} \ dS^{N-1}(x_1) \ldots dS^{N-1}(x_k)$$

$$= \int_{x_1 \in S^{N-1}} \int_{x_2 \in S^{N-2}} \ldots \int_{x_k \in S^{N-2}} \int_{\vartheta_2} \ldots \int_{\vartheta_j} \ldots \int_{\vartheta_k} dS^{N-1}(x_1)$$

$$\cdot \sin^{N-2} \vartheta_2 \ldots \sin^{N-2} \vartheta_j \ldots \sin^{N-2} \vartheta_k d\vartheta_2 \ldots d\vartheta_j dS^{N-2}(\hat{x}_2) \ldots dS^{N-2}(\hat{x}_k)$$

$$= \int_{x_1} dS^{N-1}(x_1) \int_{\vartheta_2} \ldots \int_{\vartheta_j} \ldots \int_{\vartheta_k} \sin^{N-2} \vartheta_2 \ldots \sin^{N-2} \vartheta_j \ldots \sin^{N-2} \vartheta_k d\vartheta_2 \ldots d\vartheta_j d\vartheta_k$$

$$\cdot \int_{\hat{x}_2 \in S^{N-2}} \ldots \int_{\hat{x}_k \in S^{N-2}} dS^{N-2}(\hat{x}_2) \ldots dS^{N-2}(\hat{x}_k)$$

$$= \int_{x_1} dS^{N-1}(x_1) \int_{\vartheta_2} \ldots \int_{\vartheta_j} \ldots \int_{\vartheta_k} \sin^{N-2} \vartheta_2 \ldots \sin^{N-2} \vartheta_j \ldots \sin^{N-2} \vartheta_k d\vartheta_2 \ldots d\vartheta_j d\vartheta_k$$

$$\cdot \nu(\hat{x}_2, \ldots, \hat{x}_k \in S^{N-2} : [\hat{x}_2, \ldots, \hat{x}_k] \cap B_{\rho/\sin \vartheta_1}(\hat{a}) \neq \emptyset).$$
By using the induction hypothesis for the last term (this is allowable, since \( \rho / \sin \theta_1 < |\hat{a}| \) as remarked above) and (9) we can continue:

\[
\leq \int_{x_1} dS^{N-1}(x_1) (O_{N-1}/O_{N-2})^{k-1} \cdot K(N - 1, k - 1)(\rho / \sin \theta_1)^{(N-1)-(k-1)}
\]

\[
= \left( \frac{O_{N-1}}{O_{N-2}} \right)^{k-1} K(N - 1, k - 1) \rho^{N-k} \int_{x_1 \in S^{N-1}} (1 / \sin \theta_1)^{N-k} dS^{N-1}(x_1)
\]

\[
= \left( \frac{O_{N-1}}{O_{N-2}} \right)^{k-1} K(N - 1, k - 1) \rho^{N-k} \int_{x_1 \in S^{N-2}} \int_{\theta_1} \sin \theta_1 d\theta_1 dS^{N-2}(x_1)
\]

\[
= \left( \frac{O_{N-1}}{O_{N-2}} \right)^{k-1} K(N - 1, k - 1) \rho^{N-k} \cdot O_{N-2}O_{k-1}/O_{k-2}
\]

\[
= \left( \frac{O_{N-1}}{O_{N-2}} \right)^{k-1} \frac{O_{N-2}O_{k-1}}{O_{k-2}} K(N - 1, k - 1) \rho^{N-k}.
\]

Now return to (14), and using the result of (15) we find that

\[
\nu_k(x_1, \ldots, x_k \in S^{N-1} : [x_1, \ldots, x_k] \cap B_\rho(a) \neq \emptyset)
\]

is bounded by

\[
\rho^{N-k} \left[ \rho^{k-1}(O_{N-1})^{k-1} K(N, 1) + \left( \frac{O_{N-1}}{O_{N-2}} \right)^{k-1} \frac{O_{N-2}O_{k-1}}{O_{k-2}} K(N - 1, k - 1) \right].
\]

(16)

Of course we could use \( \rho < 1 \) (thus \( \rho^{k-1} < 1 \)) and take constants \( K \) such that

\[
K(N, k) = \left[ (O_{N-1})^{k-1} K(N, 1) + \left( \frac{O_{N-1}}{O_{N-2}} \right)^{k-1} \frac{O_{N-2}O_{k-1}}{O_{k-2}} K(N - 1, k - 1) \right].
\]

Note that this is a well-defined recurrence relation as \( K(N, 1) \) is known explicitly. However, solving this recurrence relation gives a \( K(N, k) \) that increases (too) fast with \( N \). So we proceed more subtly and restrict \( \rho \) to \( (0, \alpha) \), \( \alpha \) being a constant which we shall choose as later as \( 1/\pi \). The bound in (16)
now becomes
\[
\rho^{N-k} \left[ \alpha^{k-1}(O_{N-1})^{k-1}K(N, 1) + \left( \frac{O_{N-1}}{O_{N-2}} \right)^{k-1} \frac{O_{N-2}O_{k-1}}{O_{k-2}} K(N - 1, k - 1) \right]
\]
and (as above) a possible choice for \( K(N, k) \) is the solution of
\[
K(N, k) = \alpha^{k-1}(O_{N-1})^{k-1}K(N, 1) + \left( \frac{O_{N-1}}{O_{N-2}} \right)^{k-1} \frac{O_{N-2}O_{k-1}}{O_{k-2}} K(N - 1, k - 1).
\]

All we have to do is solve this recurrence relation. By analogy with (5) we factor out the measure of the whole space, that is we divide by \( \nu(S_{N,k}) = (O_{N-1})^k \). Defining
\[
L(N, k) := \frac{K(N, k)}{(O_{N-1})^k}
\]
we get
\[
L(N, k) = \alpha^{k-1}L(N, 1) + \frac{O_{N-2}O_{k-1}}{O_{N-1}O_{k-2}} L(N - 1, k - 1).
\]  \hspace{1cm} (17)

It follows from the expression for \( K(N, 1) \), see (11), that
\[
L(N, 1) = \frac{2O_{N-2}}{(N - 1)O_{N-1}}(\pi/2)^{N-1}.
\]

The solution of (17) is simply obtained by expanding (note the nice telescope-behaviour of the second factor). We find
\[
L(N, k) = \sum_{j=1}^{k} \alpha^{k-j} \frac{O_{N-j}O_{k-1}}{O_{N-1}O_{k-j}} L(N - j + 1, 1)
\]
\[
= \frac{O_{k-1}}{O_{N-1}} \sum_{j=1}^{k} \alpha^{k-j} \frac{2O_{N-j-1}}{(N - j)O_{k-j}} \left( \frac{\pi}{2} \right)^{N-j}
\]
\[
= \frac{O_{k-1}}{O_{N-1}} \alpha^{k} \left( \frac{\pi}{2} \right)^{N} \sum_{j=1}^{k} \left( \frac{\alpha\pi}{2} \right)^{-j} \frac{2O_{N-j-1}}{(N - j)O_{k-j}}
\]
\[
= \frac{O_{k-1}}{O_{N-1}} \left( \frac{\alpha\pi}{2} \right)^{k} \left( \frac{\pi}{2} \right)^{N-k} \sum_{j=1}^{k} \left( \frac{2}{\alpha\pi} \right)^{j} \frac{2O_{N-j-1}}{(N - j)O_{k-j}}
\]

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Setting $\alpha = 1/\pi$ we get

$$L(N, k) = \frac{O_{k-1}}{O_{N-1}} \left( \frac{\pi}{2} \right)^{N-k} \left( \frac{1}{2} \right)^k \sum_{j=1}^{k} 2^j \frac{2O_{N-j-1}}{(N-j)O_{k-j}}.$$ 

Motivated by (3) we multiply and divide by $O_{N-k}/(N-k)$, to obtain

$$L(N, k) = \frac{O_{k-1}}{O_{N-1}} \left( \frac{\pi}{2} \right)^{N-k} \left( \frac{1}{2} \right)^k \sum_{j=1}^{k} 2^j \frac{2(N-k)O_{N-j-1}}{(N-j)O_{N-k-1}O_{k-j}}.$$ 

Using $O_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$ and $\Gamma(N + 1) = N\Gamma(N)$ the last fraction can be written as

$$\frac{2(N-k)O_{N-j-1}}{(N-j)O_{N-k-1}O_{k-j}} = (\pi)^{-1/2} \frac{\Gamma(\frac{k-j+1}{2})\Gamma(\frac{N-k+2}{2})}{\Gamma(\frac{N-j+2}{2})}.$$ 

This last expression is of the form $\Gamma(a)\Gamma(b)/\Gamma(a+b-\frac{1}{2})$, which can be thought of a perturbed beta function; recall that $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$. It is easily seen by the integral representation of the beta function that $B(a, b) \leq 1$; in the light of this the following lemma is unsurprising.

**Lemma 3.3**

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b - \frac{1}{2})} \leq \pi^{1/2} \quad \text{for } a, b \in \left[ \frac{1}{2}, \infty \right).$$

**Proof (Lemma 3.3).** Fix $a$; then for $b = 1/2$ we have equality. We show that

$$\frac{\Gamma(b)}{\Gamma(b + \hat{a})}, \quad \text{with } \hat{a} := a - 1/2 \geq 0 \text{ fixed},$$

is monotonically decreasing with $b$. This is equivalent to

$$\ln \Gamma(b + \hat{a}) - \ln \Gamma(b)$$
monotonically increasing with \( b \). But this is a simple consequence of the convexity of \( f := \ln \sigma \Gamma \) (see any good analysis book). As everything is smooth we require

\[
\frac{d}{dx} [f(x + \tilde{a}) - f(x)] = f'(x + \tilde{a}) - f'(x) \geq 0;
\]

replacing the difference by \( a f''(x + \theta a) \), with \( \theta \in [0, 1] \), and noting that \( f'' \) is positive (since it is smooth and convex), we obtain the result. \( \Box \).

**Proof (Theorem 3.1, conclusion).** Putting all together we have

\[
L(N, k) \leq \frac{O_{k-1}}{O_{N-1}} \left( \frac{\pi}{2} \right)^{N-k} \frac{O_{N-k-1}}{N-k} \left( \frac{1}{2} \right)^k \sum_{j=1}^{k} 2^j.
\]

Therefore a possible choice for \( K(N, k) \) is

\[
K(N, k) = 2 \cdot \nu(S_{N,k}) \frac{O_{k-1}}{O_{N-1}} \left( \frac{\pi}{2} \right)^{N-k} \frac{O_{N-k-1}}{N-k}.
\]

**4 Conclusion**

We first discuss the application of this construction to the papers by Ben-Artzi et al. [1] and Foias & Olson [3].

Using theorem 3.1 to replace the Grassmannians in the proof of [1] is straightforward; indeed (6) holds for \( 0 < \rho < |a| \) as the explicit form of \( K = K(N, k) \) is not used. A priori we have to be more careful with the proof of [3] as they use (4) for \( 0 < \rho < |a| \) (the whole range of \( \rho \)). However, Lemma 4.2 and Lemma 7.1 of [3] in fact only require (4) for \( \rho \in (0, \epsilon) \), for some \( \epsilon > 0 \). The full formula (2) is required once (Lemma 6.1), but only
for $k = 1$, which is obtained as Lemma 3.2 in this paper, the factor 2 not affecting the argument.

In these two papers the use of this new construction removes the reliance of the proofs on advanced notions of integral geometry. Furthermore, since it is much more straightforward (if tedious) to calculate the measure of a given set of planes using $S_{N,k}$ rather than the standard measure on $G_{N,k}$, we hope that our construction will prove useful in other situations where a natural first choice would seem to be the standard Grassmannian.

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