Topology of Sobolev mappings III

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Abstract

We establish a necessary and sufficient topological condition for maps in $W^{1,p}(M, N)$ to be connected by continuous paths in $W^{1,p}(M, N)$ to maps in $W^{1,q}(M, N)$, $q > p \geq 1$. We also obtain a necessary and sufficient topological condition under which $W^{1,q}(M, N)$ would be strongly dense in $W^{1,p}(M, N)$. Several results concerning sequential weak density of smooth maps (or $W^{1,q}(M, N)$ maps) in $W^{1,p}(M, N)$ are also proved.

1 Introduction

This is in sequel to [9] and [10]. To simplify the presentation, we shall always assume $M$ and $N$ are smooth compact Riemannian manifolds without boundary. In addition we assume they are isometrically embedded into $\mathbb{R}^l$ and $\mathbb{R}^l$ respectively. We shall first study the question (brought to us by H. Brezis and Y. Y. Li, see [3]) whether every map in $W^{1,p}(M, N)$ can be connected by a continuous path in $W^{1,p}(M, N)$ to a map in $W^{1,q}(M, N)$, $1 \leq p < q < \infty$. We are also interested in whether the space $W^{1,q}(M, N)$ is strongly or weakly sequentially dense in $W^{1,p}(M, N)$. It is easy to see when $q \geq n$, the space $W^{1,q}(M, N)$ can be replaced by $C^\infty(M, N)$ and in that case, these questions were already carefully studied in [3] and [10]. Indeed, in [10] we established a necessary and sufficient topological condition under which any map in $W^{1,p}(M, N)$ can be connected by a continuous path in $W^{1,p}(M, N)$ to a map in $C^\infty(M, N)$. We also showed a necessary and sufficient topological condition which guarantee the strong closure of $C^\infty(M, N)$ to be equal to $W^{1,p}(M, N)$. In addition, we found a necessary topological condition for $C^\infty(M, N)$ to be weakly sequentially dense in $W^{1,p}(M, N)$. Whether or not this topological condition is sufficient remains a challenging open problem. The main purpose of the present article is to show several partial results (see Theorem 5.5, Theorem 6.1 and Section 8) concerning this open problem.
In this paper we will first prove various results analogue to [10] for $W^{1,p}(M,N)$ (instead of $C^\infty(M,N)$), see Corollary 3.2, Theorem 4.3 and Corollary 5.2. Then we consider the weak sequential density problem. In Theorem 5.5 we establish that Sobolev maps with canonical singularities (such maps are strongly dense in $W^{1,p}(M,N)$) will be in the sequential weak closure of smooth maps whenever the above mentioned necessary topological condition is verified for such given maps. We also present a new and unified proof of some generalizations of known results concerning weak sequential density of smooth maps in Sobolev mapping spaces, in particular some results due to P. Hajlasz [7] (see Theorem 8.5 and Corollary 8.6). Finally we introduce a variational energy $I(u)$ which is naturally associated with sequential weak closure of smooth maps. A map $u \in W^{1,p}(M,N)$ is a weak sequential limit of smooth maps if and only if $I(u) < \infty$. More remarkably, if smooth maps are weakly sequentially dense in $W^{1,p}(B^n, N)$, then $I(u)$ is bounded by a constant depending only on $n, p$ and $N$ (see Theorem 9.6). The interesting connections of $I(\cdot)$ to defect measures (cf. [13]), minimal connections (cf. [2]) and, in a special case, scan (cf. [12]) are discussed in the final section of the paper.

To conclude the introduction, we would like to point out that the characterizations of whether or not a map in $W^{1,p}(M,N)$ can be connected by a continuous path in $W^{1,p}(M,N)$ to a smooth map, or whether or not a map in $W^{1,p}(M,N)$ can be weakly sequentially approximated by smooth maps lead to some natural consequences in the study of variational problems and their associated gradient flows. For example, standard arguments show that starting with a $W^{1,p}(M,N)$ map, the heat flow leads to a family $u(t)$, $0 < t < \infty$, (a weak solution) such that each $u(t)$ is $[p] - 1$ homotopic to the initial map $u(0)$, and as $t$ goes to infinity, it subconverges to limiting $p$-harmonic maps which are also $[p] - 1$ homotopic to the initial map. Similarly, one can minimize $p$-energy in a given $[p] - 1$ homotopy class to find a minimizer because the minimizing sequence in a given $[p] - 1$ homotopy class has weak limits still staying in the same $[p] - 1$ homotopy class. Different $[p] - 1$ homotopy classes obviously produce different minimizers (see [19, 20]). For the regularity of such minimizers in a given $[p] - 1$ homotopy class, we refer to discussions in [13]. Here we should not elaborate it further.

2 Some preparations

We will employ the notations introduced in Section 2 of [10]. Let us be given topological spaces $X, Y$ and $Z$ and a continuous map $\psi : X \to Y$. 
Let $\alpha \in [Y, Z]$ be a homotopy class. Then we may define $\alpha \circ \psi \in [X, Z]$ by $\alpha \circ \psi = [f \circ \psi]$ for any $f \in \alpha$. It is clear that $[f \circ \psi]$ does not depend on the specific choice of $f$ in $\alpha$. If $X$ can be endowed with some CW complex structures, $\alpha, \beta \in [X, Y], k \in \mathbb{Z}$, $k \geq 0$, such that for any $f \in \alpha$, $g \in \beta$, we have $f \sim_k g$, then we say $\alpha$ is $k$ homotopic to $\beta$.

Similar to Lemma 2.2 in [10], we have

**Lemma 2.1** Let $X$, $Y$ and $Z$ be topological spaces, $X$ and $Y$ be endowed with CW complex structures $(X^j)_{j \in \mathbb{Z}}$ and $(Y^j)_{j \in \mathbb{Z}}$ respectively, $k, l \in \mathbb{Z}$, $0 \leq k \leq l$. If $X$ is $k + 1$ homotopy equivalent to $Y$ and for any $f_0 \in C(X^k, Z)$, $f_0|_{X^k}$ has a continuous extension to $X^l$, then for any $f \in C(Y^{k+1}, Z)$, $f|_{Y^k}$ has a continuous extension to $Y^l$.

**Proof:** The argument is almost same as that for the proof of Lemma 2.2 in [10]. But for reader's convenience, we sketch it here. First we may find $\phi \in C(X, Y)$ and $\psi \in C(Y, X)$ such that $\psi \phi \sim_{k+1} id_X$ and $\phi \psi \sim_{k+1} id_Y$. By the cellular approximation theorem, we may assume both $\phi$ and $\psi$ are cellular. Given a $f \in C(Y^{k+1}, Z)$, define $f_0(x) = f(\phi(x))$ for $x \in X^{k+1}$. Then we may find a $g_0 \in C(X^l, Z)$ such that $g_0|_{X^k} = f_0|_{X^k}$.

Set $g(y) = g_0(\psi(y))$ for $y \in Y^l$. Let $i$ be the map from $Y^k$ to $Y^{k+1}$ such that $i(y) = y$ for any $y \in Y^k$, then $\phi \sim i$ as maps from $Y^k$ to $Y^{k+1}$ (see the proof of Lemma 2.2 in [10]). Hence $g|_{Y^k} \sim f|_{Y^k}$. It follows from homotopy extension property that $f|_{Y^k}$ has a continuous extension to $Y^l$.

We introduce the following

**Definition 2.2** Let $X$ and $Y$ be topological spaces. Assume $X$ can be endowed with some CW complex structures, $k, l \in \mathbb{Z}$, $0 \leq k \leq l$. If for any CW complex structure $(X^j)_{j \in \mathbb{Z}}$, any $f \in C(X^{k+1}, Y)$, $f|_{X^k}$ has a continuous extension to $X^l$, then we say $X$ satisfies the $(k, l)$ extension property with respect to $Y$.

We remark that in general $(k, l)$ extension property need not be satisfied. In fact, as in the proof of Corollary 5.5 in [10], by cohomology theory, we see $\mathbb{C}P^3$ does not satisfy $(2, 4)$ extension property with respect to $\mathbb{C}P^1$, $\mathbb{R}P^4$ does not satisfy $(1, 3)$ extension property with respect to $\mathbb{R}P^2$. 

*Note:* The above text is a partial transcription of the image, and it may contain errors due to the quality of the image. The content is presented in a natural form for reading purposes.
3 Maps in $W^{1,p}(M, N)$ which may be connected continuously to maps in $W^{1,q}(M, N)$

For $1 \leq p < \infty$, $u, v \in W^{1,p}(M, N)$, recall that $u \sim_p v$ means we may find a continuous path in $W^{1,p}(M, N)$ connecting $u$ and $v$. If $n \leq p < \infty$, it follows from [18] that every map in $W^{1,p}(M, N)$ can be connected to a smooth map. In general, this need not be the case, as we have seen in Corollary 5.5 of [10]. We will use notations and concepts in Section 3 and 4 of [10].

THEOREM 3.1 Assume $1 \leq p < q < \infty$, $p < n$, $u \in W^{1,p}(M, N)$, $h : K \to M$ is a Lipschitz rectilinear cell decomposition, then $u$ may be connected by a continuous path in $W^{1,p}(M, N)$ to a map in $W^{1,q}(M, N)$ if and only if $u_{\#} (h)$ is extendible to $|K[q]|$ with respect to $N$.

PROOF: In the case $q \geq n$, since every map in $W^{1,q}(M, N)$ may be connected continuously in $W^{1,q}(M, N)$ to a smooth map, the conclusion of the theorem follows from Proposition 5.2 in [10]. Hence we assume $1 \leq p < q < n$ below.

If for $u \in W^{1,p}(M, N)$ there is a $v \in W^{1,q}(M, N)$ with $u \sim_p v$, then it follows from Theorem 5.1 of [10] that $u_{\#} (h) = v_{\#} (h)$. Recall that if we define $h_\xi (x) = \pi (h (x) + \xi)$ for $x \in |K|$ and $\xi \in B_{\epsilon_0}^l$, here $\pi$ is the nearest point projection to $M$ and $\epsilon_0 = \epsilon_0 (M)$ is a small positive number, then for $\mathcal{H}^l$ a.e. $\xi \in B_{\epsilon_0}^l$, $v \circ h_\xi \in \mathcal{W}^{1,q} (K, N)$ and $v_{\#} (h) = \left[ v \circ h_\xi \right]_{K[q]}$.

Here

$$\mathcal{W}^{1,q} (K, N) = \left\{ f : f \in \mathcal{W}^{1,q} (K, \mathbb{R}^l) \text{ such that for each } \Delta \in K, \quad f (x) \in N \text{ for } \mathcal{H}^d \text{ a.e. } x \in \Delta, d = \dim (\Delta) \right\}.$$  

We refer the reader to Section 3 and 4 of [10] for the proof of this fact. Since $v \circ h_\xi \left|_{K[q]} \right| \in \mathcal{W}^{1,q} \left( K[q], N \right)$, it follows from Lemma 4.4 of [10] that we may find a $f \in C \left( \left| K[q] \right|, N \right)$ such that $f \left|_{K[q]-1} \right| = v \circ h_\xi \left|_{K[q]-1} \right|$. This implies in particular $f \left|_{K[q]-1} \right| = v \circ h_\xi \left|_{K[q]-1} \right|$. Hence $v_{\#} (h)$ is extendible to $|K[q]|$ with respect to $N$, so is $u_{\#} (h)$.

On the other hand, if $u_{\#} (h)$ is extendible to $|K[q]|$ with respect to $N$, from Section 2 of [10] we may find a Lipschitz map $f : |K[q]| \to N$ such that $f \left|_{K[q]-1} \right| \in u_{\#} (h)$. Now let $f_1$ be the radial extension of $f$ to
higher dimensional skeletons of $K$, and $v = f_1 \circ h^{-1}$, then $v \in W^{1,p}(M,N)$ and $v$ is $[p] - 1$ homotopic to $u$, hence it follows from Theorem 5.1 of [10] that $v \sim_p u$.

**Corollary 3.2** Assume $1 \leq p < q < \infty$, $p < n$, then every map in $W^{1,p}(M,N)$ can be connected continuously in $W^{1,p}(M,N)$ to a map in $W^{1,q}(M,N)$ if and only if $M$ satisfies the $([p] - 1, [q])$ extension property with respect to $N$.

**Proof:** If $q \geq n$, then the result follows from Corollary 5.4 of [10]. Hence we assume $1 \leq p < q < n$ below. Fix a Lipschitz triangulation of $M$, say $h : K \rightarrow M$.

If every map in $W^{1,p}(M,N)$ can be connected by a continuous path in $W^{1,p}(M,N)$ to a map in $W^{1,q}(M,N)$, then for any $f \in Lip\left(M^{[p]}, N\right)$ let $g$ be the homogeneous degree zero extension of $f \circ h\big|_{K^{[p]}}$ on higher dimensional skeletons to $|K|$, then $u = g \circ h^{-1} \in W^{1,p}(M,N)$ and $u_{\#_p}(h) = \left[f \circ h\big|_{K^{[p-1]}}\right]$. Since $u$ can be connected continuously in $W^{1,p}(M,N)$ to a map in $W^{1,q}(M,N)$, from Theorem 3.1 we know $f \circ h\big|_{K^{[p-1]}}$ has a continuous extension to $|K^{[q]}|$. Hence $f|_{K^{[p-1]}}$ has a continuous extension to $M^{[q]}$. By Proposition 2.3 in [10] and homotopy extension property we know $M$ satisfies the $([p] - 1, [q])$ extension property with respect to $N$.

On the other hand, if $M$ satisfies the $([p] - 1, [q])$ extension property with respect to $N$, then given any $u \in W^{1,p}(M,N)$, after going through a continuous path in $W^{1,p}(M,N)$, we may assume there exists a $\xi \in B_{\varepsilon_0}$ such that $u \circ h\xi\big|_{K^{[p]}} \in Lip\left(|K^{[p]}|, N\right)$ and $u_{\#_p}(h) = \left[u \circ h\xi\big|_{K^{[p-1]}}\right]$. Hence by Theorem 3.1, $u$ may be connected by a continuous path in $W^{1,p}(M,N)$ to a map in $W^{1,q}(M,N)$.

**4 Strong density problems**

Given $1 \leq p < q < \infty$, we may define

$$H^{1,p}_{S,q}(M,N) = \overline{W^{1,q}(M,N)} \text{ in } W^{1,p}(M,N),$$

and

$$H^{1,p}_{W,q}(M,N) = \left\{ u : u \in W^{1,p}(M,N), \text{ there exists a sequence} \right\}$$

$$u_i \in W^{1,q}(M,N) \text{ such that } u_i \rightharpoonup u \text{ in } W^{1,p}(M,R^d) \right\}. $$
Clearly we have $H^1_{S,q} (M, N) \subset H^1_{W,q} (M, N) \subset W^{1,p} (M, N)$. When $q \geq n$, we have $H^1_{S,q} (M, N) = H^1_p (M, N)$ and $H^1_{W,q} (M, N) = H^1_p (M, N)$ (see [9]). The following theorem, which generalizes Theorem 6.2 of [10], plays a crucial role in the future development.

**Theorem 4.1** Assume $1 \leq p < q < \infty$, $p < n$, $h : K \to M$ is a Lipschitz rectilinear cell decomposition, $M^i = h ([K^i])$ for $i \geq 0$, $L^{n-[p]-1}$ is a dual $n-[p]-1$ skeleton of $K$, $u \in W^{1,p} (M, N)$ is continuous on $M \setminus h \left(L^{n-[p]-1}\right)$, then $u \in H^1_{S,q} (M, N)$ if and only if $u_{|_{M^d}}$ has a continuous extension to $M^d$.

**Proof:** If $q \geq n$, then the theorem follows from Theorem 6.2 of [10], hence we assume $q < n$ below.

Assume $u \in H^1_{S,q} (M, N)$. If $[p] = [q]$, then clearly $u_{|_{M^d}}$ has a continuous extension to $M^d$. Hence we assume $[p] < [q]$ below. By the definition of $H^1_{S,q} (M, N)$, we may find a sequence $u_i \in W^{1,q} (M, N)$ such that $u_i \to u$ in $W^{1,p} (M, N)$. Let $H (x, \xi) = \pi (h (x) + \xi)$ for $x \in |K|$ and $\xi \in B_{\varepsilon_0}$, here $\varepsilon_0 = \varepsilon_0 (M)$ is a small positive number and $\pi$ is the nearest point projection to $M$. Then we may find an $\varepsilon_1 > 0$ such that $\chi_{[p],H,u} (\xi) = \left[ u \circ h_{|[K^p]} \right]$ $\mathcal{H}^i$ a.e. on $B_{\varepsilon_1}$. It follows from Proposition 4.1 of [10] that after passing to a subsequence, we have $\chi_{[p],H,u,i} \to \chi_{[p],H,u}$ $\mathcal{H}^i$ a.e. on $B_{\varepsilon_1}$. Hence we may find a $\xi \in B_{\varepsilon_1}$ such that

$$u_i \circ h_{\xi} \in W^{1,q} (K, N), \chi_{[p],H,u,i} (\xi) = \left[ u_i \circ h_{\xi} \right]_{K^p},$$

$$u \circ h_{\xi} \in W^{1,q} (K, N), \chi_{[p],H,u} (\xi) = \left[ u \circ h_{\xi} \right]_{K^p} = \left[ u \circ h \right]_{K^p} = u_{|_{K^p}}$$

for $i$ large enough.

On the other hand, fix an $i'$ large enough, it follows from Lemma 4.4 of [10] that we may find a $f \in C \left(|K^d|, N\right)$ such that $f_{|_{K^{d-1}}} = u_{i'} \circ h_{\xi}$. Hence $f_{|_{K^p}} = u_{i'} \circ h_{\xi}$. This implies $u_{i'} \circ h_{\xi}$ has a continuous extension to $|K^p|$, so is $u \circ h_{|_{K^p}}$ in view of homotopy extension property. Put it in another way, $u_{|_{M^d}}$ has a continuous extension to $M^d$.

Now assume $u_{|_{M^d}}$ has a continuous extension to $M^d$, we want to show $u \in H^1_{S,q} (M, N)$. In view of Lemma 6.1 and Proposition 2.2 of [10], we may assume $u$ is smooth on $M \setminus h \left(L^{n-[p]-1}\right)$. 

We will use the notations in Section 6 of [10]. For convenience, denote \( k = [p] \) and \( \bar{k} = [q] \). also we write \( \Gamma_\epsilon, \phi \) and \( F_{\delta,\epsilon} \) instead of \( \Gamma_\epsilon^k, \phi^k \) and \( F_{\delta,\epsilon}^k \). Let \( f = u \circ h \), then \( f|_{\bar{K}^k} \) has a continuous extension to \( \bar{K}^\bar{k} \), let us call it \( \bar{f} \). For \( x \in \bar{K}^\bar{k} \cup \{ x \in |K| : |x|_k \geq \epsilon \} \), we set \( \bar{f}(x) = \bar{f} (F_{\epsilon,1} (x)) \).

Then we claim that \( \bar{f} \) is homotopic to \( f \) on \( \{ x \in |K| : |x|_k \geq \epsilon \} \). In fact if \( \epsilon \leq t \leq 1, x \in |K|, |x|_k \geq \epsilon, \) we set \( h_t (x) = f (F_{t,1} (x)) \), then \( h_t \) is the needed homotopy. By homotopy extension property we know \( f \) has a continuous extension to \( [K] \cup \{ x \in |K| : |x| \geq \epsilon \} \), say \( g \). It follows from Section 2 of [10] and the fact \( f \) is Lipschitz that we may assume \( g \) is Lipschitz. Now we want to extend \( g \) to the whole \( |K| \) such that it is in \( W^{1,q} (K, N) \). This can be done by induction. Pick any \( \Delta \in K \) with \( \dim \Delta = \bar{k} + 1 \), it follows from the definition of \( |x|_k \) that \( \{ x \in \Delta : |x|_k \leq \epsilon \} \) is starshaped with respect to the point \( y_\Delta \) (see Section 6 of [10]). Hence we may define \( g \) on the whole \( \Delta \) by homogeneously degree zero extension. Then we do the similar construction on every \( \bar{k} + 2 \) dimensional cell. The procedure finishes after we process all \( n \) dimensional cells. Clearly \( g \) is \( W^{1,q} (K, N) \).

For \( 0 < \delta \leq \epsilon \leq \frac{1}{2} \), we set \( f_{\delta,\epsilon} (x) = g(F_{\delta,\epsilon} (x)) \) for \( x \in |K| \), then \( f_{\delta,\epsilon} \in W^{1,q} (K, N) \). It is clear that \( \{ x \in |K| : f_{\delta,\epsilon} (x) \neq f (x) \} \subset \{ x \in |K| : |x|_k \leq \epsilon \} \). Hence to estimate \( |f_{\delta,\epsilon} - f|_{W^{1,p} (K)} \) we only need to give an upper bound for \( \int_{|x|_k \leq \epsilon} |df_{\delta,\epsilon} (x)|^p \, d\mathcal{H}^n (x) \). To do this we shall need to refer to the properties of the deformations with respect to the dual skeleton \((P_1)\)\(\cdots\)\((P_5)\) in the proof of Theorem 6.2 in [10].

At first we have the following computation

\[
\int_{|x|_k \leq \delta} |df_{\delta,\epsilon} (x)|^p \, d\mathcal{H}^n (x) \\
\leq c(p, K) \epsilon^p \delta^{-p} \int_{|x|_k \leq \delta} |dg(F_{\delta,\epsilon} (x))|^p \, d\mathcal{H}^n (x) \quad \text{(by (P4))}
\]

\[
\leq c(p, K) \epsilon^p \delta^{-p} \int_{|x|_k \leq \delta} |dg(F_{\delta,\epsilon} (x))|^p J_{|x|_k} (x) \, d\mathcal{H}^n (x) \quad \text{(by (P2))}
\]

\[
= c(p, K) \epsilon^p \delta^{-p} \int_0^\delta \int_{|x|_k = r} |dg\left(\phi_{\epsilon,x} (x)\right)|^p \, d\mathcal{H}^{n-1} (x)
\]

\[
= c(p, K) \epsilon^p \delta^{-p} \int_0^\delta \int_{|y|_k = \frac{r}{\epsilon}} |dg\left(y\right)|^p J\left(\phi_{\epsilon,x}\left(y\right)\right) \, d\mathcal{H}^{n-1} (y)
\]
Here we have used the change of variable formula. It follows that

\[
\int_{|x|_h \leq \delta} |df_{\delta, \varepsilon}(x)|^p \, d\mathcal{H}^n(x) \\
\leq c(p, K) \varepsilon^{p-k} \delta^{k-p} \int_0^\delta \, dr \int_{|y|_h = \frac{r}{\varepsilon}} |dg(y)|^p \, d\mathcal{H}^{n-1}(y) \quad \text{(by (P_5))}
\]

\[
= c(p, K) \varepsilon^{p-k-1} \delta^{k+1-p} \int_0^\varepsilon \, dr \int_{|y|_h = r} |dg(y)|^p \, d\mathcal{H}^{n-1}(y)
\]

\[
\leq c(p, K) \varepsilon^{p-k-1} \delta^{k+1-p} \int_{|y|_h \leq \varepsilon} |dg(y)|^p \, d\mathcal{H}^n(y).
\]

We use (P_2) and the coarea formula in the last inequality. The same argument as in the proof of Theorem 6.2 of [10] shows

\[
\int_{\delta \leq |x|_h \leq \varepsilon} |df_{\delta, \varepsilon}(x)|^p \, d\mathcal{H}^n(x) \leq c(p, K) \varepsilon \int_{\Gamma_\varepsilon} |df|^p \, d\mathcal{H}^{n-1},
\]

and for \(0 < t \leq \frac{1}{2}\), we may find an \(\varepsilon_t \in [t, 2t]\) such that

\[
\varepsilon_t \int_{\Gamma_\varepsilon_t} |df|^p \, d\mathcal{H}^{n-1} \leq c(p, K) \int_{t \leq |x|_h \leq 2t} |df|^p \, d\mathcal{H}^n \to 0 \quad \text{as} \ t \to 0^+.
\]

Hence

\[
|f_{\delta, \varepsilon t} - f|_{W^{1,p}(K)} \leq \alpha_1(\delta, t) + \alpha_2(t),
\]

where \(\alpha_1(\delta, t) \to 0\) if we fix \(t\) and let \(\delta \to 0^+\), \(\alpha_2(t) \to 0\) as \(t \to 0^+\).

Observing that \(u_{\delta, \varepsilon t} = f_{\delta, \varepsilon t} \circ h^{-1} \in W^{1,q}(\mathcal{M}, N)\) and

\[
|u_{\delta, \varepsilon t} - u|_{W^{1,p}(\mathcal{M})} \leq c(p, M) |f_{\delta, \varepsilon t} - f|_{W^{1,p}(K)},
\]

we see \(u \in H^{1,p}_{S_i,q}(\mathcal{M}, N)\). 

Before stating the next theorem, we need the following lemma, the proof of which goes exactly the same as the proof of Lemma 6.5 in [10].

**Lemma 4.2** Let \(X\) and \(Y\) be two topological spaces. Assume \(X\) can be equipped with some CW complex structures, \(Y\) is path connected, \(k \in \mathbb{N}\), \(l \in \mathbb{Z}\), \(k \leq l\), \(\pi_k(Y) = 0\). Then \(X\) satisfies the \((k-1, l)\) extension property with respect to \(Y\) if and only if for any CW complex decomposition of \(X\), namely \((X^i)_{i \in \mathbb{Z}}\), any \(f \in L^1(X^k, Y)\), we may find a \(g \in C(X^l, Y)\) such that \(g|_{X^k} = f\). It is also equivalent to the condition that \(X\) satisfies the \((k, l)\) extension property with respect to \(Y\).
THEOREM 4.3 Assume $1 \leq p < q < \infty$, $p < n$. If $[p] = [q]$, then we always have $H^1_{S,q} (M, N) = W^{1,p} (M, N)$. If $[p] < [q]$, then $H^1_{S,q} (M, N) = W^{1,p} (M, N)$ if and only if $\pi_{[p]} (N) = 0$ and $M$ satisfies the $(|p| - 1, [q])$ extension property with respect to $N$.

PROOF: If $q \geq n$, then the conclusion follows from Theorem 6.3 of [10]. Hence we assume $q < n$ below.

Assume $[p] = [q]$. It follows from Theorem 4.1 that for any $u \in R^{p,\infty} (M, N)$ (see Section 6 of [10] for definition), we have $u \in H^1_{S,q} (M, N)$. Hence $R^{p,\infty} (M, N) \subset H^1_{S,q} (M, N)$. On the other hand, it follows from Theorem 6.1 of [10] that $R^{p,\infty} (M, N) = W^{1,p} (M, N)$, hence we have $H^1_{S,q} (M, N) = W^{1,p} (M, N)$.

From now on, we assume $[p] < [q]$. If $H^1_{S,q} (M, N) = W^{1,p} (M, N)$, then pick up a smooth triangulation of $M$, namely $h : K \to M$, denote $M^i = h ([K^i])$ for $i \geq 0$. For each $\Delta \in K$, choose a point $y_\Delta \in \text{int} \Delta$. Given any $f \in \text{Lip} \left( M^{[p]}, N \right)$, define $f_0 = f \circ h$. Let $f_1 \in W^{1,p} (K, N)$ be the map which we get from $f_0$ by doing homogeneous degree zero extension with respect to $y_\Delta$ on all simplices $\Delta$ with $\text{dim} \Delta \geq [p] + 1$, then $u = f_1 \circ h^{-1} \in W^{1,p} (M, N)$. Hence $u \in H^1_{S,q} (M, N)$. It follows from Theorem 4.1 that $u|_{M^{[p]}} = f$ has a continuous extension to $M^{[q]}$. It then follows from Proposition 2.3 of [10] and homotopy extension property that for any $f \in C \left( M^{[p]}, N \right)$, $f$ has a continuous extension to $M^{[q]}$. Clearly this implies $\pi_{[p]} (N) = 0$ and $M$ satisfies the $(|p| - 1, [q])$ extension property with respect to $N$.

On the other hand, if $\pi_{[p]} (N) = 0$ and $M$ satisfies the $(|p| - 1, [q])$ extension property with respect to $N$, then it follows from Lemma 4.2 that for any CW complex decomposition of $M$, any $f \in C \left( M^{[p]}, N \right)$, $f$ has a continuous extension to $M^{[q]}$. In view of Theorem 6.1 in [10], we only need to show $R^{p,\infty} (M, N) \subset H^1_{S,q} (M, N)$, but this clearly follows from the topological statement above and Theorem 4.1.

COROLLARY 4.4 Assume $N$ is connected, $1 \leq p < q < n$, if $\pi_i (N) = 0$ for $[p] \leq i \leq [q] - 1$, then $H^1_{S,q} (M, N) = W^{1,p} (M, N)$.

PROOF: This follows from Theorem 4.3 and cell by cell extension.
5 Weak sequential density problem

We now turn to the question of whether or not $W^{1,q}(M, N)$ maps are weakly sequentially dense in $W^{1,p}(M, N)$. First of all, we have the following necessary condition for a map in $W^{1,p}(M, N)$ to be a sequential weak limit of maps in $W^{1,q}(M, N)$.

**Theorem 5.1** Assume $1 \leq p < q < \infty$, $p < n$, $u \in W^{1,p}(M, N)$, $h : K \to M$ is a Lipschitz rectilinear cell decomposition of $M$. If $u \in H_{W,q}^{1,p}(M, N)$, then $u_{\#}^p (h)$ is extendible to $[K^{[q]}]$ with respect to $N$, hence $u$ may be connected to a map in $W^{1,q}(M, N)$ by a continuous path in $W^{1,p}(M, N)$.

**Proof:** If $q \geq n$, then the theorem follows from Theorem 7.1 in [10]. Hence we assume $q < n$ below. Observe that from the proof of Theorem 3.1, for any $v \in W^{1,q}(M, N)$, $v_{\#}^q (h)$ is extendible to $[K^{[q]}]$. Hence Theorem 5.1 follows from Theorem 3.1 and an earlier result of B. White in [20] (see also Proposition 4.1 of [10]).

An immediate consequence of the above statement is the following:

**Corollary 5.2** Assume $1 \leq p < q < \infty$, $p < n$. If $H_{W,q}^{1,p}(M, N) = W^{1,p}(M, N)$, then $M$ satisfies the $(\lceil p \rceil - 1, [q])$ extension property with respect to $N$.

**Proof:** This follows from Corollary 3.2 and Theorem 5.1.

In some cases, the set of all weak sequential limits of $W^{1,q}(M, N)$ maps in $W^{1,p}(M, N)$ is equal to the strong closure of $W^{1,q}(M, N)$.

**Theorem 5.3** If $p = 1$ or $1 < p < n$ but $p \notin Z$ or $2 \leq p < n$, $p \in Z$ but $\pi_p (N) = 0$, then for any $p < q < \infty$, we have $H_{W,q}^{1,p}(M, N) = H_{W,q}^{1,p}(M, N)$.

**Proof:** The case $p = 1$ follows from the similar argument as the proof of Theorem 1.1 in [8]. We only need to replace Theorem 6.2 of [10] by Theorem 4.1. The case $1 < p < n$, $p \notin Z$ follows from the proof of Theorem 6.1 in [10] and Theorem 4.1 (one should refer to Remark 6.1 of [10]). The case $2 \leq p < n, p \in Z$, $\pi_p (N) = 0$ follows from the proof of Theorem 6.1 in [10] and Theorem 4.1 (one should refer to the proof of Theorem 7.2 in [10]).
Now we shall give another proof of Theorem 6.2 in [10]. The proof will also facilitate the proof of the main result of this section, Theorem 5.5 below.

**THEOREM 5.4 (Theorem 6.2 in [10])** Assume $1 \leq p < n$, $h : K \to M$ is a Lipschitz rectilinear cell decomposition. $M^i = h \left( \left[ K^i \right] \right)$ for $i \geq 0$, $L^{n-[p]-1}$ is a dual $n-[p]-1$ skeleton, $u \in W^{1,p}(M,N)$ is continuous on $M \setminus h \left( L^{n-[p]-1} \right)$, then $u \in H_0^{1,p}(M,N)$ if and only if $u|_{M^0}$ has a continuous extension to $M$. In addition, if for some $\alpha \in [M,N]$, we have $u|_{M^0} \in \alpha|_{M^0}$, then we may find a sequence $u_i \in C^\infty(M,N)$ such that $[u_i] = \alpha$ and $u_i \to u$ in $W^{1,p}(M,N)$.

**PROOF:** The necessary part can be done in the same way as in the original proof of Theorem 6.2 in [10].

Let us look at the sufficient part. For convenience we shall use the notations in Section 6 of [10]. First in view of Lemma 6.1 in [10], we may assume $u$ is smooth on $M \setminus h \left( L^{n-[p]-1} \right)$. Let $k = [p]$. We shall write $\phi$ instead $\phi^k$. Choose a $v \in C(M,N)$ such that $[v] = \alpha$. Let $g_0 = v \circ h$, $f = u \circ h$. We claim there exists a $g \in Lip([K],N)$ such that $g = f$ on $\left\{ x \in [K] : |x|_k \geq \frac{1}{2} \right\}$ and $g \sim g_0$. Indeed, since $\left\{ x \in [K] : |x|_k \geq \frac{1}{4} \right\}$ has $[K]^k$ as a deformation retractor, it follows from $f|_{[K]^k} \sim g_0|_{[K]^k}$ that $f \sim g_0$ on $\left\{ x \in [K] : |x|_k \geq \frac{1}{4} \right\}$. This plus the homotopy extension theorem and approximation by Lipschitz maps imply that we may find a Lipschitz extension of $f|_{\left\{ x \in [K] : |x|_k \geq \frac{1}{4} \right\}}$ which is homotopic to $g_0$. Now for $0 < \varepsilon \leq \frac{1}{2}$, we may define

$$f_\varepsilon(x) = \begin{cases} 
  f(x), & \varepsilon \leq |x|_k; \\
  f\left( \phi_{\frac{|x|_k}{\varepsilon \ln \delta}}(x) \right), & 2\varepsilon^2 \leq |x|_k \leq \varepsilon; \\
  g\left( \phi_{\frac{|x|_k}{2\varepsilon^2}}(x) \right), & 0 < |x|_k \leq 2\varepsilon^2; \\
  g(x), & |x|_k = 0.
\end{cases}$$

Clearly $f_\varepsilon \sim g$. In addition, we have the following inequalities: we are still using the co-area formula and the properties $(P_1) - (P_5)$ in the proof of Theorem 6.2 of [10] and estimates in Appendix B of [10], but to keep the proof short we will go without mention.
\[
\int_{2\varepsilon^2 \leq |x|_k \leq \varepsilon} |df_\varepsilon(x)|^p \, dH^n(x)
\]
\[
\leq c(p, K) \int_{2\varepsilon^2 \leq |x|_k \leq \varepsilon} \left| df\left(\frac{\phi}{|x|_k^2}\right)(x)\right|^p \left(\frac{\varepsilon^2}{|x|_k^2}\right)^p \, dH^n(x)
\]
\[
\leq c(p, K) \int_{2\varepsilon^2 \leq |x|_k \leq \varepsilon} \left| df\left(\frac{\phi}{|x|_k^2}\right)(x)\right|^p \left(\frac{\varepsilon^2}{|x|_k^2}\right)^p J_{|x|_k}(x) \, dH^n(x)
\]
\[
= c(p, K) \int_{2\varepsilon^2}^{\varepsilon} dr \int_{|x|_k = r} \left| df\left(\frac{\phi}{r^2}\right)(x)\right|^p \left(\frac{\varepsilon^2}{r^2}\right)^{2p-2k} \, dH^{n-1}(x)
\]
\[
\leq c(p, K) \int_{2\varepsilon^2}^{\varepsilon} dr \int_{|x|_k = r} \left| df(x)\right|^p \left(\frac{\varepsilon}{r}\right)^{2p-2k} \, dH^{n-1}(x)
\]
\[
= c(p, K) \int_{\varepsilon}^{2\varepsilon^2} \left(\frac{\varepsilon}{r}\right)^{2k+2p} \left(\int_{|x|_k = r} \left| df(x)\right|^p \, dH^{n-1}(x)\right) \, dr,
\]
we observe that the last term above tends to 0 as \(\varepsilon \to 0^+\) by Lebesgue’s dominated convergence theorem. We also have

\[
\int_{|x|_k \leq 2\varepsilon^2} |df_\varepsilon(x)|^p \, dH^n(x)
\]
\[
\leq c(p, K) \int_{|x|_k \leq 2\varepsilon^2} \left| d\left(\phi_{|x|_k^2}\right)(x)\right|^p \left(\frac{1}{\varepsilon^2}\right)^p \, dH^n(x)
\]
\[
\leq c(p, K) \int_{0}^{2\varepsilon^2} dr \int_{|x|_k = r} \left| d\left(\phi_{r^2}\right)(x)\right|^p \left(\frac{1}{\varepsilon^2}\right)^p \, dH^{n-1}(x)
\]
\[
\leq c(p, K) \int_{0}^{2\varepsilon^2} dr \int_{|x|_k = \frac{r}{\varepsilon^2}} \left| d\left(\phi_{\frac{r}{\varepsilon^2}}(x)\right)\right|^p \, d\varepsilon \, dH^{n-1}(x)
\]
\[
= c(p, K) \varepsilon^{2k+2p-2p} \int_{0}^{\frac{\varepsilon}{r}} dr \int_{|x|_k = r} \left| d\left(\phi_{\frac{r}{\varepsilon^2}}(x)\right)\right|^p \, dH^{n-1}(x)
\]
\[
\leq c(p, K) \varepsilon^{2k+2p-2p} \int_{|x|_k \leq \frac{1}{\varepsilon^2}} \left| d\left(\phi_{\frac{r}{\varepsilon^2}}(x)\right)\right|^p \, dH^n(x) \to 0 \quad \text{as} \quad \varepsilon \to 0^+.
\]

The above two series of inequalities imply in particular that

\[
\int_{|x|_k \leq \varepsilon} |df_\varepsilon(x)|^p \, dH^n(x) \to 0 \quad \text{as} \quad \varepsilon \to 0^+,
\]

it follows from this and the definition of \(f_\varepsilon\) that \(\int_{|K|} |df_\varepsilon - df|^p \, dH^n \to 0\) as \(\varepsilon \to 0^+\). If we denote \(u_\varepsilon = f_\varepsilon \circ h^{-1}\), then \(u_\varepsilon \in Lip(M, N)\), \([u_\varepsilon] = \alpha\) and \(u_\varepsilon \to u\) in \(W^{1,p}(M, N)\) as \(\varepsilon \to 0^+\). Theorem 5.4 follows. \(\blacksquare\)
The proof of the following theorem follows from the similar line of arguments above. Note that we are able to preserve the additional topological informations in choosing the weakly convergent sequence.

**Theorem 5.5** Assume $2 \leq p < n$, $p \in \mathbb{Z}$, $h : K \to M$ is a Lipschitz rectilinear cell decomposition. $M^i = h \left( |K^i| \right)$ for $i \geq 0$, $L^{n-p-1}$ is a dual $n-p-1$ skeleton, $u \in W^{1,p}(M,N)$ such that $u$ is continuous on $M \setminus h(L^{n-p-1})$, then $u \in H_W^{1,p}(M,N)$ if and only if $u|_{MP-1}$ has a continuous extension to $M$. In addition, if for some $\alpha \in [M,N]$, we have $u|_{MP-1} \in \alpha|_{MP-1}$, then we may find a sequence $u_i \in C^\infty(M,N)$ such that $u_i = \alpha$, $u_i \to u$ in $W^{1,p}(M,N)$ and $du_i \to du$ a.e. on $M$.

**Proof:** It is clear that for such $u$, $u_{#,p}(h) = \left[ u \circ h \right]_{[K^{p-1}]}$. If $u \in H_W^{1,p}(M,N)$, then it follows from Theorem 7.1 in [10] that $u_{#,p}(h)$ has a continuous extension to $M$, this implies that $u \circ h|_{[K^{p-1}]}$ has a continuous extension to $|K|$, hence $u|_{MP-1}$ has a continuous extension to $M$.

On the other hand, given $u$ as in the theorem such that for some $\alpha \in [M,N]$, $u|_{MP-1} \in \alpha|_{MP-1}$, we want to construct a sequence $u_i \in C^\infty(M,N)$ with $u_i = \alpha$ and $u_i \to u$ in $W^{1,p}(M,N)$. Again we will use the notations in Section 6 of [10]. Let $k = p-1$, we shall write $\phi, \Gamma_\epsilon$ and $F_{\delta,\epsilon}$ instead of $\phi^k, \Gamma_\epsilon^k$ and $F_{\delta,\epsilon}^k$. Pick up a $v \in \alpha$, let $g_0 = v \circ h$, $f = u \circ h$, then we claim that there exists a $g \in C(|K|,N) \cap W^{1,p}(K,N)$ such that $g = f$ on $\left\{ x \in |K| : |x|_k \geq \frac{1}{2} \right\}$ and $g \sim g_0$.

To prove the claim, we first show that there exists a $g_1 \in C(|K|,N)$ such that $g_1\left|_{\left\{ x \in |K| : |x|_k \geq \frac{1}{4} \right\}} = f\left|_{\left\{ x \in |K| : |x|_k \geq \frac{1}{4} \right\}} \right.$ and $g_1 \sim g_0$. In fact since $f\left|_{|K^k|} \sim g_0\left|_{|K^k|} \right.$, by homotopy extension theorem we may find a $\tilde{f} \in C(|K|,N)$ such that $\tilde{f}\left|_{|K^k|} = f\left|_{|K^k|} \right.$ and $\tilde{f} \sim g_0$. Now let $\tilde{f}(x) = \tilde{f}\left( F_{\frac{1}{10},1}(x) \right)$ for $x \in |K|$, then $\tilde{f} \in C(|K|,N)$ and $\tilde{f} \sim \tilde{f} \sim g_0$.

We observe that $\tilde{f}\left|_{\left\{ x \in |K| : |x|_k \geq \frac{1}{4} \right\}}$ is homotopic to $f\left|_{\left\{ x \in |K| : |x|_k \geq \frac{1}{4} \right\}}$. In fact, $u_4(x) = f(F_{\frac{1}{10},1}(x))$ for $\frac{1}{10} \leq t \leq 1$, $x \in |K|$, $|x|_k \geq \frac{1}{10}$ is a homotopy between them. It follows again from homotopy extension theorem that we may find the needed $g_1$. Let $\varepsilon_0 = \varepsilon_0(N)$ be a small positive number such that $V_{2\varepsilon_0}(N) = \left\{ x \in \mathbb{R}^P : \text{dist}(x,N) < 2\varepsilon_0 \right\}$ is a tubular neighborhood of $N$ and let $\pi : V_{2\varepsilon_0}(N) \to N$ be the nearest point projection. From Proposition 2.3 of [10] we may find a $g_2 \in Lip(M,N)$ such that $\|g_2 - g_1\|_{L^\infty(M)} \leq \varepsilon_0$, hence $g_2 \sim g_1$. Now let $g_3(x) = \eta(|x|_k) f(x) + (1 - \eta(|x|_k)) g_2(x)$ for $x \in |K|$. Here $\eta \in C^\infty([0,1], \mathbb{R})$, $0 \leq \eta \leq 1$, $\eta|_{[\frac{1}{3},1]} = 1$ and $\eta = 0$ on
\[0, \frac{1}{8}\]. Then \( g_3 \in C \left( |K|, \mathbb{R}^r \right) \cap \overline{W}^{1,p} \left( K, \mathbb{R}^r \right) \) and \( |g_3(x) - g_2(x)| \leq \varepsilon_0 \) for any \( x \in |K| \). Hence if we set \( g = \pi \circ g_3 \), then \( g \) satisfies all the requirements in the claim.

Now for \( 0 < \varepsilon \leq \frac{1}{2} \), we may define \( f_\varepsilon : |K| \rightarrow N \) by

\[
f_\varepsilon(x) = \begin{cases} f(x), & \varepsilon \leq |x|_k; \\ f \left( \phi_{\frac{|x|_k}{\varepsilon}}(x) \right), & 2\varepsilon^2 \leq |x|_k \leq \varepsilon; \\ g \left( \phi_{\frac{|x|_k}{\varepsilon^2}}(x) \right), & 0 < |x|_k \leq 2\varepsilon^2; \\ g(x), & |x|_k = 0. \end{cases}
\]

Then \( f_\varepsilon \in C(\{K\}, N) \cap \overline{W}^{1,p}(K, N) \), \( f_\varepsilon \sim g \) and the proof for Theorem 5.4 shows

\[
\int_{2\varepsilon^2 \leq |x|_k \leq \varepsilon} |df_\varepsilon(x)|^p d\mathcal{H}^n(x) \leq c(p, K) \int_{\varepsilon \leq |x|_k \leq \frac{1}{2}} |df(x)|^p d\mathcal{H}^n(x),
\]

\[
\int_{|x|_k \leq 2\varepsilon^2} |df_\varepsilon(x)|^p d\mathcal{H}^n(x) \leq c(p, K) \int_{|x|_k \leq \frac{1}{2}} |dg(x)|^p d\mathcal{H}^n(x).
\]

Hence

\[
\int_{|K|} |df_\varepsilon|^p d\mathcal{H}^n \leq c(p, K) \int_{|K|} |df|^p d\mathcal{H}^n + c(p, K) \int_{|x|_k \leq \frac{1}{2}} |dg(x)|^p d\mathcal{H}^n(x).
\]

If we define \( u_\varepsilon = f_\varepsilon \circ h^{-1} \), then clearly we have \( u_\varepsilon \rightarrow u \) in \( L^p(M, N) \) and

(5.1) \[
\int_M |du_\varepsilon|^p d\mathcal{H}^n \leq c(p, h, K) \int_M |du|^p d\mathcal{H}^n + c(p, h, K) \int_{|x|_k \leq \frac{1}{2}} |dg(x)|^p d\mathcal{H}^n(x).
\]

Hence we may find a sequence \( \varepsilon_i \rightarrow 0^+ \) such that \( u_{\varepsilon_i} \rightarrow u \) in \( W^{1,p}(M, N) \). On the other hand, since \( u_{\varepsilon_i} \in C(M, N) \cap W^{1,p}(M, N) \), \( |u_{\varepsilon_i}| = \alpha \), and

\[
\mathcal{H}^n (\{u_{\varepsilon_i} \neq u\}) \leq c(M) \varepsilon_i^p,
\]

Theorem 5.5 follows.

Let \( 2 \leq p < n, p \in \mathbb{Z} \). If we define \( \overline{H}^{1,p}_M(M, N) \) as the smallest subset of \( W^{1,p}(M, N) \) which contains \( C^\infty(M, N) \) and is closed under the sequential weak convergence in \( W^{1,p}(M, N) \), then it follows from Chapter 3, Section 4.1 of [6] that \( \overline{H}^{1,p}_M(M, N) \) is equal to the successive sequential limits of
$C^\infty(M, N)$ in $W^{1,p}(M, N)$ up to the first uncountable ordinal number. In fact it follows from Proposition 4.1, Theorem 6.1, Theorem 7.2 in [10] and Theorem 5.5 that if we fix a Lipschitz rectilinear cell decomposition of $M$, namely $h : K \to M$, then

$$
\tilde{H}_W^{1,p}(M, N)
= \left\{ u : u \in W^{1,p}(M, N), \, u_{\#_p}(h) \text{ is extendible to } M \text{ w.r.t. } N \right\},
$$

and $\tilde{H}_W^{1,p}(M, N) = \overline{H}_W^{1,p}(M, N)$, here the closure is taken under the strong topology of $W^{1,p}(M, N)$. In particular, we only need second time sequential weak limits to get $\tilde{H}_W^{1,p}(M, N)$ from $C^\infty(M, N)$ instead of going up to the first uncountable ordinal number. The Conjecture 7.1 of [10] just asks whether $\tilde{H}_W^{1,p}(M, N) = H_W^{1,p}(M, N)$. In Section 8, we shall prove this is the case when $N$ satisfies certain topological condition. Similar to Theorem 5.5, we have

**Theorem 5.6** Assume $2 \leq p < n$, $p \in \mathbb{Z}$, $p < q < \infty$, $h : K \to M$ is a Lipschitz rectilinear cell decomposition. $M^i = h([K^i])$ for $i \geq 0$, $L_n^{n-p-1}$ is a dual $n-p-1$ skeleton, $u \in W^{1,p}(M, N)$ such that $u$ is continuous on $M \setminus h(L_n^{n-p-1})$, then $u \in H_W^{1,p}(M, N)$ if and only if $u|_{M^{n-p-1}}$ has a continuous extension to $M^q$. Indeed, if $u|_{M^{n-p-1}}$ has a continuous extension to $M^q$, then we may find a sequence $u_i \in W^{1,q}(M, N)$ such that $u_i \rightharpoonup u$ in $W^{1,p}(M, N)$ and $du_i \rightarrow du$ a.e. on $M$.

**Proof:** The necessary part follows from Theorem 5.1. The sufficient part follows from the same proof of Theorem 5.5.

6 Weak sequential convergence of critical power Sobolev mappings

Recall it follows from [18] or [4] that for any $u \in W^{1,n}(M, N)$, we may associate with it a homotopy class in $[M, N]$, which we still denote as $[u]$ (we remind the reader that $n = \dim M$). Fix an $\alpha \in [M, N]$, we denote

$$
W_\alpha^{1,n}(M, N) = \left\{ u \in W^{1,n}(M, N) : [u] = \alpha \right\}.
$$

Because the homotopy class corresponding to a $W^{1,n}(M, N)$ map is stable under perturbation in the strong topology, we easily see

$$
W_\alpha^{1,n}(M, N) = \left\{ u \in C^\infty(M, N) : [u] = \alpha \right\}
$$
and \( W^{1,n}_{\alpha} (M, N) \) is both an open and closed subset of \( W^{1,n} (M, N) \). In fact it is one of the (path) connected components in \( W^{1,n} (M, N) \). Let \( u \in W^{1,n} (M, N) \), \( K \) be a finite rectilinar cell complex, \( h : |K| \to M \) be a Lipschitz map, then it follows from Proposition 4.1 in [10] and approximation by smooth maps that \( [u] \circ (h|_{K^{n-1}}) = u_{\neq, n}(h) \). For \( n \geq 2 \), we are interested in the following

\[
H^{1,n}_{\alpha, W} (M, N) = \left\{ u \in W^{1,n} (M, N) : \text{there exists a sequence of maps} \right. \\
u_i \in C^\infty (M, N) \text{ with } [u_i] = \alpha \text{ and } u_i \to u \text{ in } W^{1,n} \left. \right\}.
\]

**Theorem 6.1** Assume \( n \geq 2 \), \( \alpha \in [M, N] \), then

\[
H^{1,n}_{\alpha, W} (M, N) = \left\{ u \in W^{1,n} (M, N) : [u] \text{ is } n - 1 \text{ homotopic to } \alpha \right\}.
\]

In addition, for any \( u \in H^{1,n}_{\alpha, W} (M, N) \), we may find a sequence \( u_i \in C^\infty (M, N) \) such that \( [u_i] = \alpha \), \( u_i \to u \) in \( W^{1,n} (M, N) \) and \( du_i \to du \) a.e. on \( M \).

To prove this theorem, we need the following technical lemma.

**Lemma 6.2** Let \( X \) be a \( m \) dimensional smooth Riemannian manifold without boundary, \( \Omega \subset X \) be an open subset with compact closure and Lipschitz boundary. Assume \( u \in W^{1,m} (\Omega, N) \) such that the trace \( T_{\partial \Omega} (u) = f \in W^{1,m} (\partial \Omega, N) \subset C (\partial \Omega, N) \), then for any \( \varepsilon > 0 \), there exists a \( v \in W^{1,m} (\Omega, N) \cap C (\overline{\Omega}, N) \) such that \( |v - u|_{W^{1,m} (\Omega)} \leq \varepsilon \) and \( v|_{\partial \Omega} = f \). In addition, there exists an \( \varepsilon = \varepsilon (u, \Omega, N) > 0 \) such that for any \( v_1, v_2 \in W^{1,m} (\Omega, N) \cap C (\overline{\Omega}, N) \), with \( |v_i - u|_{W^{1,m} (\Omega)} \leq \varepsilon \) and \( v_i|_{\partial \Omega} = f \) for \( i = 1, 2 \), then we have \( v_1 \sim v_2 \) relative to \( \partial \Omega \), that is, during the homotopy, the value on \( \partial \Omega \) is always fixed.

The proof of Lemma 6.2 is just a slight modification of the proof of Lemma 4.4 in [10]. Let \( [\overline{\Omega}, \partial \Omega; N] \) be the set of all continuous maps from \( \overline{\Omega} \) to \( N \) modulus the equivalence of homotopy relative to \( \partial \Omega \), then it clearly follows from Lemma 6.2 that for any \( u \in W^{1,m} (\Omega, N) \) with \( T_{\partial \Omega} (u) \in W^{1,m} (\partial \Omega, N) \), there is an element in \( [\overline{\Omega}, \partial \Omega; N] \) naturally corresponding to \( u \), which we denote \([u, \partial \Omega] \).

**Proof of Theorem 6.1:** If \( u \in H^{1,n}_{\alpha, W} (M, N) \), then we may find a sequence \( u_i \in C^\infty (M, N) \) with \([u_i] = \alpha \) and \( u_i \to u \) in \( W^{1,n} (M, N) \). It
follows from [20] or Theorem 4.1 in [10] that for $i$ large enough, $u_i$ is $n - 1$ homotopic to $u$, this implies that $\alpha = [u_i]$ is $n - 1$ homotopic to $[u]$.

Now assume $u \in W^{1,n}(M,N)$ such that $[u]$ is $n - 1$ homotopic to $\alpha$. Fix a Lipschitz triangulation of $M$, namely $h : K \to M$. For each $\Delta \in K$ with dim $\Delta = n$, pick up a point $y_{\Delta} \in \text{Int}_\Delta$, then for any $x \in \Delta$, we have the Minkowski norm $|x|_{\Delta} = \inf\{\lambda : \lambda > 0, x \in y_{\Delta} + \lambda (\Delta - y_{\Delta})\}$. As in the proof of Theorem 5.5, we write $\phi$ and $\Gamma_{\varepsilon}$ instead of $\phi^{n-1}$ and $\Gamma_{\varepsilon}^{n-1}$. Let $\varepsilon_0 = \varepsilon_0 (M)$ be a small positive number such that $V_{2\varepsilon_0} (M) = \{x : x \in \mathbb{R}^n, \text{dist} (x, M) < 2\varepsilon_0\}$ is a tubular neighborhood of $M$. Denote $\pi : V_{2\varepsilon_0} (M) \to M$ as the nearest point projection. For $\xi \in B_1^{j} \varepsilon_0$, we have $h_{\xi} (x) = \pi (h (x) + \xi)$ for $x \in |K|$, then it follows from Section 3 of [10] that for $H^l$ a.e. $\xi \in B_1^{j} \varepsilon_0$, $u \circ h_{\xi} \in W^{1,n} (K,N)$ and $[u \circ h_{\xi}|_{|K|^{n-1}}] = u_{\#_{n}} (h_{\xi})$. Choosing a small but generic $\xi$ and replacing $h$ by $h_{\xi}$, we may assume $f = u \circ h \in W^{1,n} (K,N)$ and $[u \circ h|_{|K|^{n-1}}] = u_{\#_{n}} (h)$. Choose a $v \in \alpha$, denote $g_1 = v \circ h$. It follows from Section 3 of [10] that we may choose a $\theta \in \left(\frac{1}{3}, \frac{1}{2}\right)$ such that for any $\Delta \in K$ with dim $\Delta = n$, we have $f|_{\{x \in \Delta : |x|_\Delta = \theta\}} \in W^{1,n} (\{x \in \Delta : |x|_\Delta = \theta\}, N)$. For each $\Delta \in K$ with dim $\Delta = n$, let $\Omega_\Delta = \{x \in \Delta : \theta < |x|_\Delta < 1\}$. Then it follows from Lemma 6.2 that we may find a $\tilde{f} \in W^{1,n} (|K|, N) \cap C (|K|, N)$ such that for any $\Delta \in K$ with dim $\Delta = n$, $[f|_{\Omega_\Delta}, \partial \Omega_\Delta] = [\tilde{f}|_{\Omega_\Delta}, \partial \Omega_\Delta]$. We claim that there exists a $g \in C (|K|, N) \cap W^{1,n} (K,N)$ such that $g \sim g_1$ and $g|_{\Omega_\Delta} = \tilde{f}|_{\Omega_\Delta}$ for any $\Delta \in K$ with dim $\Delta = n$. Indeed, since $[u]$ is $n - 1$ homotopic to $\alpha$, we have $f|_{|K|^{n-1}}$ is homotopic to $g_1|_{|K|^{n-1}}$, this clearly implies $\tilde{f}|_{\{x \in |K|^{n-1} : \frac{1}{2} \leq |x|_{|K|^{n-1}} \leq 1\}}$ is homotopic to $g_1|_{\{x \in |K|^{n-1} : \frac{1}{2} \leq |x|_{|K|^{n-1}} \leq 1\}}$. It follows from homotopy extension property that we may find a $g_2 \in C (|K|, N)$ such that $g_2|_{\{x \in |K|^{n-1} : \frac{1}{2} \leq |x|_{|K|^{n-1}} \leq 1\}} = \tilde{f}|_{\{x \in |K|^{n-1} : \frac{1}{2} \leq |x|_{|K|^{n-1}} \leq 1\}}$ and $g_2 \sim g_1$. Let $\varepsilon_0 = \varepsilon_0 (N)$ be a small positive number such that $V_{2\varepsilon_0} (N) = \{x \in \mathbb{R}^n : \text{dist} (x, N) < 2\varepsilon_0\}$ is a tubular neighborhood of $N$. Denote $\pi : V_{2\varepsilon_0} (N) \to N$ as the nearest point projection. From Section 2 of [10], we may find a $g_3 \in Lip (|K|, N)$ such that $|g_3 - g_2|_{L^\infty (|K|)} \leq \varepsilon_0$. Now let $g_4 (x) = \eta \left(|x|_{|K|^{n-1}}\right) \tilde{f} (x) + \left(1 - \eta \left(|x|_{|K|^{n-1}}\right)\right) g_3 (x)$ for $x \in |K|$. Here $\eta \in C^\infty ([0,1], \mathbb{R})$, $0 \leq \eta \leq 1$, $\eta|_{[0,\frac{1}{2}]} = 1$, $\eta|_{[\frac{1}{2},1]} = 0$. Clearly $g_4 \in W^{1,n} (K) \cap C (|K|)$ and for any $x \in |K|$, dist$(g_4 (x), N) \leq \varepsilon_0$. Now $g = \pi \circ g_4$ is the needed map for the claim.
Given $0 < \varepsilon \leq \frac{1}{3}$, we define $f_\varepsilon$ by

$$f_\varepsilon (x) = \begin{cases} f (x), & \varepsilon \leq |x|_{n-1}; \\ f \left( \frac{\phi_{\frac{1}{10}} (x)}{\varepsilon^2} \right), & \frac{\varepsilon^2}{\theta} \leq |x|_{n-1} \leq \varepsilon; \\ g \left( \frac{\phi_{\frac{1}{10}} (x)}{\varepsilon^2} \right), & 0 < |x|_{n-1} \leq \frac{\varepsilon^2}{\theta}; \\ g (x), & |x|_{n-1} = 0. \end{cases}$$

Then $f_\varepsilon \in \tilde{W}^{1,n} (K, N)$ and $f_\varepsilon \to f$ in $L^n (|K|)$,

$$\int_{|K|} |df|^n d\mathcal{H}^n \leq c (K) \int_{|K|} |df|^n d\mathcal{H}^n + c (K) \int_{|x|_{n-1} \leq \theta} |dg (x)|^n d\mathcal{H}^n (x).$$

If we set $u_\varepsilon = f_\varepsilon \circ h^{-1}$, then clearly we may find a sequence $\varepsilon_i \to 0^+$ such that $u_\varepsilon \to u$ in $\tilde{W}^{1,n} (M, N)$. To complete the proof of Theorem 6.1, we only need to show $u_\varepsilon \in W^{1,n}_\partial (M, N)$. Since $f_\varepsilon$ may be connected to $f_\theta$ in $\tilde{W}^{1,n} (|K|, N)$, we know $u_\varepsilon$ can be connected to $u_\theta$. Hence we only need to show $[u_\theta] = \alpha$. By Lemma 6.2, for any $\delta > 0$, we may find a $\tilde{f}_\delta \in \tilde{W}^{1,n} \left( \{ x \in |K| : \theta \leq |x|_{n-1} \leq 1 \}, N \right)$ such that $\tilde{f}_\delta$ is continuous, for any $\Delta \in K$ with $\dim \Delta = n$, $[\tilde{f}_\delta |_{\Omega_\Delta}, \partial \Omega_\Delta] = [f |_{\Omega_\Delta}, \partial \Omega_\Delta]$ and $|\tilde{f}_\delta - f|_{\tilde{W}^{1,n} (\{ x \in |K| : \theta \leq |x|_{n-1} \leq 1 \})} \leq \delta$. Now let

$$f_{\theta, \delta} (x) = \begin{cases} \tilde{f}_\delta (x), & \theta \leq |x|_{n-1} \leq 1; \\ g (x), & |x|_{n-1} \leq \theta. \end{cases}$$

Then $f_{\theta, \delta} \in \tilde{W}^{1,n} (K, N) \cap C (|K|, N)$ and $f_{\theta, \delta} \to f_\theta$ in $\tilde{W}^{1,n} (K, N)$. If we set $u_{\theta, \delta} = f_{\theta, \delta} \circ h^{-1}$, then $u_{\theta, \delta} \in \tilde{W}^{1,n} (M, N) \cap C (M, N)$, $[u_{\theta, \delta}] = \alpha$ and $u_{\theta, \delta} \to u_\theta$ as $\delta \to 0^+$, hence $[u_\theta] = \alpha$. The last assertion of Theorem 6.1 clearly follows from the construction.

### 7 Generic composition of Sobolev mappings

Assume $\Omega \subset \mathbb{R}^n$ is a bounded open subset with Lipschitz boundary, $1 \leq p < \infty$, $u \in \tilde{W}^{1,p} (\Omega, \mathbb{R}^m)$, $f \in \tilde{W}^{1,p}_{loc} (\mathbb{R}^m, \mathbb{R})$. If either $f$ or $u$ has better regularity, then we may talk about the composition $f \circ u$ such that $f \circ u \in \tilde{W}^{1,p} (\Omega, \mathbb{R})$. Indeed, it is clear that if $u \in \tilde{W}^{1,p} (\Omega, \mathbb{R}^m)$, $f \in C^1 (\mathbb{R}^m, \mathbb{R})$ and $\sup_{y \in \mathbb{R}^m} |\nabla f (y)| < \infty$, then $f \circ u \in \tilde{W}^{1,p} (\Omega, \mathbb{R})$ and

$$d (f \circ u)_x = df_{u(x)} \circ du_x \quad \text{for } \mathcal{H}^n \text{ a.e. } x \in \Omega.$$
When $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $f \in \text{Lip} (\mathbb{R}^m, \mathbb{R})$, it is easy to see that we still have $f \circ u \in W^{1,p}(\Omega, \mathbb{R})$, but the equation (7.1) need not be correct because $u$ may map a positive measure set into a set where $f$ is not differentiable. Nevertheless, it was proved in [1, 14, 15] that if for $x \in \Omega$, we denote $T_{u,x} = u(x) + du_x (\mathbb{R}^m)$, then for $\mathcal{H}^m$ a.e. $x \in \Omega$, $f|_{T_{u,x}}$ is differentiable at $u(x)$ and

\begin{equation}
    d (f \circ u)_{x} = d \left( f|_{T_{u,x}} \right)_{u(x)} \circ du_x.
\end{equation}

On the other hand, if we know $m = n$, $f \in W^{1,p}_{\alpha} (\mathbb{R}^m, \mathbb{R})$ and $u$ is a bi-Lipschitz map from $\Omega$ to $u(\Omega)$, then we also have $f \circ u \in W^{1,p}(\Omega, \mathbb{R})$ and the equation (7.1) remains true. If we do not have better regularity for $f$ or $u$, the usual composition is impossible. But we will show below that if we perturb $u$ a little bit, then in some sense, the composition is still fine. This type of arguments first appeared in [11], where it was used to construct comparison maps for variational problems.

Let us describe the setup. One should compare with Section 3 of [10]. We always assume in this section that $M$ is a $n$ dimensional Riemannian manifold without boundary, $\Omega \subset M$ is a domain with compact closure and Lipschitz boundary. Assume the parameter space $P$ is a $m$ dimensional Riemannian manifold, $Q$ is a $d$ dimensional Riemannian manifold without boundary and $D \subset Q$ is a domain with compact closure and Lipschitz boundary. Assume $m \geq n$, $1 \leq p < \infty$.

Given a Borel map $H : \overline{D} \times P \to M$, we write $H(x, \xi) = H_{\xi}(x) = H^x(\xi)$. We introduce the following notations for conditions on $H$.

(C$_1$) There exists a positive constant $c_0$ such that for $\mathcal{H}^d$ a.e. $x \in \overline{D}$, we have $[H^x]_{\text{Lip}(P)} \leq c_0$.

(C$_2$) There exists a positive constant $c_1$ such that for $\mathcal{H}^d$ a.e. $x \in \overline{D}$, the Jacobian $J_{H^x}(\xi) \geq c_1$ for $\mathcal{H}^m$ a.e. $\xi \in P$.

(C$_3$) There exists a positive constant $c_2$ such that for $\mathcal{H}^d$ a.e. $x \in \overline{D}$, we have $\mathcal{H}^{m-n} \left( (H^x)^{-1}(y) \right) \leq c_2$ for $\mathcal{H}^m$ a.e. $y \in M$.

(C$_4$) For $\mathcal{H}^m$ a.e. $\xi \in P$, we have $H_{\xi} \in W^{1,p}(D,M)$. In addition, there exists a $g \in L^p(D)$ such that for $\mathcal{H}^m$ a.e. $\xi \in P$, we have $|dH_{\xi}(x)| \leq g(x)$ for $\mathcal{H}^d$ a.e. $x \in D$. 


Lemma 7.1  Given a Borel map $H : \overline{D} \times P \to M$ satisfying $(C_1), (C_2)$ and $(C_3)$, then we have for any Borel function $\chi : M \to \mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$ with $\chi \geq 0$,

$$\int_P d\mathcal{H}^m (\xi) \int_D \chi (H^x (x)) d\mathcal{H}^d (x) \leq c_1^{-1} c_2 \mathcal{H}^d (D) \int_M \chi (y) d\mathcal{H}^m (y).$$

In particular for any Borel set $E \subset M$, we have

$$\int_P \mathcal{H}^d \left( H^{-1}_\xi (E) \right) d\mathcal{H}^m (\xi) \leq c_1^{-1} c_2 \mathcal{H}^d (D) \mathcal{H}^m (E).$$

If in addition $\mathcal{H}^m (E) = 0$, then $\mathcal{H}^d \left( H^{-1}_\xi (E) \right) = 0$ for $\mathcal{H}^m$ a.e. $\xi \in P$.

Proof:  By the co-area formula, we have

$$\int_P d\mathcal{H}^m (\xi) \int_D \chi (H^x (x)) d\mathcal{H}^d (x)$$

$$= \int_D d\mathcal{H}^d (x) \int_P \chi (H^x (x)) d\mathcal{H}^m (\xi)$$

$$\leq c_1^{-1} \int_D d\mathcal{H}^d (x) \int_P \chi (H^x (x)) J_{H^x} (\xi) d\mathcal{H}^m (\xi)$$

$$= c_1^{-1} \int_D d\mathcal{H}^d (x) \int_M \chi (y) J_{H^x} (\xi) d\mathcal{H}^m (y)$$

$$\leq c_1^{-1} c_2 \mathcal{H}^d (D) \int_M \chi (y) d\mathcal{H}^m (y).$$

\[
\]

Corollary 7.2  Given a Borel map $H : \overline{D} \times P \to \overline{M} \subset M$ satisfying $(C_1), (C_2)$ and $(C_3)$, $1 \leq p < \infty$, $f \in L^p (\Omega)$, then for $\mathcal{H}^m$ a.e. $\xi \in P$, we have $f \circ H_\xi \in L^p (D)$ and

$$\int_P d\mathcal{H}^m (\xi) \int_D |f (H^x (x))| d\mathcal{H}^d (x) \leq c_1^{-1} c_2 \mathcal{H}^d (D) \| f \|_{L^p (\Omega)}^p.$$ 

If $f_i \in L^p (\Omega)$ satisfies $f_i \to f$ in $L^p (\Omega)$, then after passing to a subsequence, $f_i \circ H_\xi \to f \circ H_\xi$ in $L^p (D)$ for $\mathcal{H}^m$ a.e. $\xi \in P$.

Proof:  The first assertion follows from Lemma 7.1 by setting

$$\chi (y) = \begin{cases} 
|f (y)|^p, & \text{for } y \in \Omega \\
0, & \text{for } y \notin \Omega 
\end{cases}.$$
For the second assertion, we only need to observe that

\[ \int_P d\mathcal{H}^m(\xi) \int_D |f_i (H_\xi(x)) - f (H_\xi(x))|^p d\mathcal{H}^d(x) \]
\[ \leq c_1^{-1} c_2 \mathcal{H}^d(D) |f_i - f|^p_{L^p(\Omega)} \to 0. \]

Hence after passing to a subsequence, we have

\[ \int_D |f_\xi (H_\xi(x)) - f (H_\xi(x))|^p d\mathcal{H}^d(x) \to 0 \quad \text{for } \mathcal{H}^m \text{ a.e. } \xi \in P. \]

\[ \square \]

**Proposition 7.3** Assume $1 \leq p < \infty$, $H : \overline{\Omega} \times P \to \overline{\Omega} \subset M$ is a Borel map satisfying $(C_1), (C_2), (C_3)$ and $(C_4)$, $f \in W^{1,p}(\Omega, \mathbb{R})$, then we have the following statements.

1. **There exists a Borel set $E \subset P$ such that $\mathcal{H}^m(E) = 0$ and for any $\xi \in P \setminus E$, we have**
   
   (i) $f \circ H_\xi \in W^{1,p}(D)$.
   
   (ii) $f$ is approximately differentiable at $H_\xi(x)$ for $\mathcal{H}^d$ a.e. $x \in D$, in addition
   
   \[ d^{ap}(f \circ H_\xi)_x = d^{ap}f_{H_\xi(x)} \circ d^{ap}(H_\xi)_x \quad \text{for } \mathcal{H}^d \text{ a.e. } x \in D. \]

2. **If we define $\tilde{f}(\xi) = f \circ H_\xi$ for $\xi \in P$, then $\tilde{f} \in L^p(P, W^{1,p}(D))$. In addition, we have**

\[ \int_P d\mathcal{H}^m(\xi) \int_D \left| d(f \circ H_\xi)_x \right|^p d\mathcal{H}^d(x) \leq c_1^{-1} c_2 |d|_{L^p(\Omega)}^p |g|_{L^p(D)}^p. \]

**Proof:** Choose a Borel set $X_0 \subset \Omega$ such that $\mathcal{H}^n(X_0) = 0$ and for any $x \in \Omega \setminus X_0$, $d^{ap}f(x)$ exists. For $x \in X_0$, we may simply set $d^{ap}f(x) = 0$.

Choose a sequence $f_i \in C^\infty(M, \mathbb{R})$ such that $f_i|_{\Omega} \to f$ in $W^{1,p}(\Omega, \mathbb{R})$. We may find a Borel set $E_0 \subset P$ such that $\mathcal{H}^m(E_0) = 0$ and for $\xi \in P \setminus E_0$, $H_\xi \in W^{1,p}(D)$. From Lemma 7.1 we may find a Borel set $E_1 \subset P$ such that $\mathcal{H}^m(E_1) = 0$ and $\mathcal{H}^d(H_\xi^{-1}(X_0)) = 0$ for any $\xi \in P \setminus E_1$. On the
other hand, it follows from Corollary 7.2 and
\[
\int_P d\mathcal{H}^m (\xi) \int_D \left| \left( df_i \right) H_{\xi}(x) \circ d (H_{\xi})_x - (d^{op} f)_{H_{\xi}(x)} \circ d (H_{\xi})_x \right|^p d\mathcal{H}^d (x)
\leq \int_P d\mathcal{H}^m (\xi) \int_D \left| g (x) \right|^p \left| \left( df_i \right) H_{\xi}(x) - (d^{op} f)_{H_{\xi}(x)} \right|^p d\mathcal{H}^d (x)
\leq c_1^{-1} \int_D d\mathcal{H}^d (x) \int_P \left| g (x) \right|^p \left| \left( df_i \right) H_{\xi}(x) - (d^{op} f)_{H_{\xi}(x)} \right|^p \mathcal{H}^{m-n} \left((H^x)^{-1} (y)\right) d\mathcal{H}^n (y)
= c_1^{-1} \int_D d\mathcal{H}^d (x) \left| \int_\Omega \left| \left( df_i \right) y - (d^{op} f)_y \right|^p \mathcal{H}^{m-n} (\mathcal{H}^x)^{-1} (y) \right| d\mathcal{H}^n (y)
\leq c_1^{-1} c_2 |g|_{L^p(D)}^p |df_i - d^{op} f|_{L^p(\Omega)}^p \to 0 \quad \text{as } i \to \infty,
\]
that we may find a subsequence \( f_{i'} \) and a Borel set \( E_2 \subset P \) such that \( \mathcal{H}^m (E_2) = 0 \) and for any \( \xi \in P \setminus E_2 \), we have
\[
\int_D \left| f_{i'} (H_{\xi}(x)) - f (H_{\xi}(x)) \right|^p d\mathcal{H}^d (x) \to 0,
\]
\[
\int_D \left| \left( df_i \right) H_{\xi}(x) \circ d (H_{\xi})_x - (d^{op} f)_{H_{\xi}(x)} \circ d (H_{\xi})_x \right|^p d\mathcal{H}^d (x) \to 0.
\]
Then for \( \xi \in P \setminus (E_0 \cup E_1 \cup E_2) \), we have \( f_{i'} \circ H_{\xi} \to f \circ H_{\xi} \) in \( L^p (D) \), \( f \) is approximately differentiable at \( H_{\xi}(x) \) for \( \mathcal{H}^d \) a.e. \( x \in D \) and \( (df_{i'})_{H_{\xi}(x)} \circ dH_{\xi}(\cdot) \to d^{op} f(\cdot) \circ dH_{\xi}(\cdot) \) in \( L^p (D) \). Since \( f_{i'} \circ H_{\xi} \in W^{1,p} (D, \mathbb{R}) \) and \( d (f_{i'} \circ H_{\xi})(\cdot) = (df_{i'})|_{H_{\xi}(\cdot)} \circ dH_{\xi}(\cdot) \), we know \( f \circ H_{\xi} \in W^{1,p} (D) \) and \( d^{op} (f \circ H_{\xi})_x = d^{op} f_{H_{\xi}(x)} \circ d (H_{\xi})_x \) for \( \mathcal{H}^d \) a.e. \( x \in D \). Clearly \( f_{i'} \to f \) \( \mathcal{H}^m \) a.e. on \( P \), hence \( f \) is measurable. The two inequalities in the second assertion follow from similar computations above.

**Corollary 7.4** Given \( H \) and \( p \) the same as in Proposition 7.3, assume \( f_i, f \in W^{1,p} (\Omega) \) are such that \( f_i \to f \) in \( W^{1,p} (\Omega) \), then after passing to a subsequence we have \( f_{i'} \circ H_{\xi} \to f \circ H_{\xi} \) in \( W^{1,p} (D) \), for \( \mathcal{H}^m \) a.e. \( \xi \in P \).

**Proof:** This follows from Corollary 7.2 and
\[
\int_P d\mathcal{H}^m (\xi) \int_D \left| d (f_{i'} \circ H_{\xi})_x - d (f \circ H_{\xi})_x \right|^p d\mathcal{H}^d (x)
\leq c_1^{-1} c_2 |g|_{L^p(D)}^p |df_i - df|_{L^p(\Omega)}^p \to 0 \quad \text{as } i \to \infty.
\]

\[\square\]
8 Density problems revisited

In this section we will apply the methods in Section 7 to density problems. We shall prove a weak sequential density result which generalizes the earlier theorem by Hajlasz in [7] (see also [16]).

**Lemma 8.1** Assume $N$ is connected, $1 \leq p < \dim N$, then there exist a sequence $u_i \in C^\infty(N, N)$ such that $u_i \to \text{id}_N$ in $W^{1,p}(N, N)$ and $u_i \sim \text{const}$ if and only if $N$ is $[p]$ connected.

**Proof:** Choose $\varepsilon_0 = \varepsilon_0(N) > 0$ small such that $V_{2\varepsilon_0}(N) = \{ y \in \mathbb{R}^N : d(y, N) < 2\varepsilon_0 \}$ is a tubular neighborhood of $N$. Let $\pi$ be the nearest point projection. Choose a Lipschitz triangulation $h : K \to N$. Denote $N^i = h([K^i])$ for $0 \leq i \leq \dim N$, $H(x, \xi) = \pi(h(x) + \xi)$ for $x \in |K|$, $\xi \in B^N_{\varepsilon_0}$.

Assume we may find a sequence $u_i \in C^\infty(N, N)$ such that $u_i \to \text{id}_N$ in $W^{1,p}(N, N)$ and $u_i \sim \text{const}$, then it follows from Proposition 4.1 in [10] (see also [19]) that after passing to a subsequence $x_{[p], h, u_i} \to x_{[p], h, \text{id}_N}$. Since it is clear that $x_{[p], h, u_i} = [u_i \circ h|_{K[p]}] = [\text{const}]$, $x_{[p], h, \text{id}_N} = [h|_{K[p]}]$, we have $[h|_{K[p]}] = [\text{const}]$. Hence the identity map $i : N[p] \to N$ is homotopic to constant map. Because $i_* : \pi_j \left(N[p]\right) \to \pi_j(N)$ is always onto for $1 \leq j \leq [p]$, we see $N$ is $[p]$ connected.

On the other hand, if we know $N$ is $[p]$ connected, then it follows from induction and homotopy extension theorem that $\text{id}|_{N[p]} \in [\text{const}]|_{N[p]}$ (see [19]), it follows from Theorem 5.4 (see also [19]) that we may find a sequence $u_i \in C^\infty(N, N)$ such that $u_i \to \text{id}_N$ in $W^{1,p}(N, N)$ and $u_i \sim \text{const}$.

**Lemma 8.2** Assume $N$ is connected, $2 \leq p \leq \dim N$, $p \in \mathbb{Z}$, then there exists a sequence $u_i \in C^\infty(N, N)$ such that $u_i \to \text{id}_N$ in $W^{1,p}(N, N)$ and $u_i \sim \text{const}$ if and only if $N$ is $p - 1$ connected. In addition, if $N$ is $p - 1$ connected, then we may find a sequence $u_i \in C^\infty(N, N)$ such that $u_i \sim \text{const}$, $u_i \to \text{id}_N$ in $W^{1,p}(N, N)$ and $d(u_i)_x \to \text{id}_{N_x}$ for a.e. $x \in N$.

**Proof:** Fix a Lipschitz triangulation of $N$, namely $h : K \to N$.

If there exists a sequence $u_i \in C^\infty(M, N)$ such that $u_i \to \text{id}_N$ in $W^{1,p}(N, N)$ and $u_i \sim \text{const}$, it follows from [20] (see also Theorem 4.1 in [10]) that $(u_i)_#(h) = (\text{id}_N)_#(h)$ for $i$ large enough. This implies $[h|_{K[p-1]}] = [\text{const}]$. Hence the identity map $i : N^{p-1} \to N$ is homotopic.
to constant map, and this implies $N$ is $p-1$ connected by the same argument as in the proof of Lemma 8.1.

On the other hand, if $N$ is $p-1$ connected, then again we have $\id_N|_{N^{p-1}} \in [\text{const}]|_{N^{p-1}}$. Using Theorem 5.5 for the case $p<\dim N$ and Theorem 6.1 for the case $p=\dim N$ we may find $u_i \in C^\infty(M,N)$ such that $u_i \to \id_N$ in $W^{1,p}(N,N)$, $u_i \sim \text{const}$ and $d(u_i)_x \to d\id_{N_x}$ for a.e. $x \in N$.

The following weak sequential density result was first proved in [7], later it was proved in [16] by a different method. We shall use the point of view in Section 7 to give a new proof.

**Proposition 8.3** If $2 \leq p < n$, $p \in \mathbb{Z}$, $N$ is $p-1$ connected, then $H^p_W(M,N) = W^{1,p}(M,N)$. In fact, for any $u \in W^{1,p}(M,N)$, we may find a sequence $u_i \in C^\infty(M,N)$ such that $u_i \to u$ in $W^{1,p}(M,N)$, $|du_i|_{L^p(M)} \leq c(p,M,N)|du|_{L^p(M)}$ and $u_i \to u$ a.e. on $M$.

**Proof:** Choose $\varepsilon_0 = \varepsilon_0(N) > 0$ small such that $V_{2\varepsilon_0}(N) = \{y \in \mathbb{R}^n : d(y,N) < 2\varepsilon_0\}$ is a tubular neighborhood of $N$. Let $\pi$ be the nearest point projection. For $\xi \in B^\varepsilon_{\varepsilon_0}$, we write $\phi_\xi(y) = \pi(y + \xi)$ for $y \in N$. We may assume $\varepsilon_0$ is small enough such that $\phi_\xi$ is a diffeomorphism from $N$ to itself with $|d(\phi_\xi^{-1})_{(y)}| \leq c(N)$ on $N$.

Since $N$ is $p-1$ connected, we clearly have $p \leq m = \dim N$. It follows from Lemma 8.2 that we may find a sequence of $u_i \in C^\infty(M,N)$ such that $v_i \to \id_N$ in $W^{1,p}(N,N)$, $v_i \sim \text{const}$ and $dv_i \to d(\id_N)$ a.e. on $N$. Let $\mathcal{B}$ be a Borel set in $N$ such that $\mathcal{H}^m(\mathcal{B}) = 0$ and $d(v_i)_y \to d\id_{N_y}$ for $y \in N \setminus \mathcal{B}$. In view of Theorem 6.1 in [10], we only need to deal with maps in $R^{p,\infty}(M,N)$. Given any $u \in R^{p,\infty}(M,N)$, let $h: K \to M$ be the Lipschitz triangulation of $M$, $L^{n-p-1}$ be one of the $n-p-1$ dual skeleton such that $u$ is smooth on $M \setminus h(L^{n-p-1})$, $M^i = h(K^i)$ for $0 \leq i \leq n$.

We may define $H: M \times B^\varepsilon_{\varepsilon_0} \to N$ by $H(x,\xi) = \pi(u(x) + \xi) = \phi_\xi(u(x))$. It follows from Proposition 7.3 that

$$
\int_{B^\varepsilon_0} dH^\perp(\xi) \int_M \left| d(v_i \circ \phi_\xi \circ u)_{x} \right|^p d\mathcal{H}^n(x)
\leq c(p,M,N)|du|_{L^p(M)}^p |dv_i|_{L^p(N)}^p
\leq c(p,M,N)|du|_{L^p(M)}^p.
$$

Note here we use the fact that we may take $g(x) = c(N)|du_x|$ in condition $(C_4)$. Hence for $H^\perp$ a.e. $\xi \in B^\varepsilon_{\varepsilon_0}$, we may find a subsequence $v_{i_\eta}$
such that \( \sup \|d(v_i \circ \phi \circ u)\|_{L^p(M)} < \infty \). On the other hand, it follows from Lemma 7.1, Corollary 7.2 and Proposition 7.3 that after passing to a subsequence we have for \( \mathcal{H}^r \) a.e. \( \xi \in B_{\varepsilon_0}^r \), \( v_{\xi} \circ \phi \circ u \to \phi \circ u \) in \( L^p(M) \), and \( \mathcal{H}^n \left( H^{-1}_\xi (B) \right) = 0 \), hence we may find a \( \xi \in B_{\varepsilon_0}^r \) and a subsequence \( v_{\xi'} \) such that \( \sup \|d(v_{\xi'} \circ \phi \circ u)\|_{L^p(M)} < \infty \), \( v_{\xi'} \circ \phi \circ u \to \phi \circ u \) in \( L^p(M) \) and \( d(v_{\xi'} \circ \phi \circ u) \to d(\phi \circ u) \) a.e. on \( M \). Since \( \phi^{-1}_\xi \) is a smooth map, we clearly have \( \phi^{-1}_\xi \circ v_{\xi'} \circ \phi \circ u \to u \) in \( W^{1,p} (M,N) \) and \( d \left( \phi^{-1}_\xi \circ v_{\xi'} \circ \phi \circ u \right) \to du \) a.e. on \( M \). On the other hand, since \( v_{\xi'} \sim \text{const} \), we know \( \phi^{-1}_\xi \circ v_{\xi'} \circ \phi \circ u \mid_{M^p} \) is also homotopic to constant map, hence it has a continuous extension to \( M \) by the homotopy extension theorem. It follows from Theorem 6.2 in [10] that \( \phi^{-1}_\xi \circ v_{\xi'} \circ \phi \circ u \in H^{1,p}_S (M,N) \). Hence \( u \in H^{1,p}_S (M,N) \). To get a sequence of smooth maps weakly convergent to \( u \) together with energy bound and a.e. convergent differentials, we observe that from Corollary 7.2 we have

\[
\int_{B_{\varepsilon_0}^r} d\mathcal{H}^r(\xi) \int_M |v_i \circ \phi \circ u - \phi \circ u|^p d\mathcal{H}^n(x) \leq c(p,M,N) |v_i - i d\mathcal{N}|_{L^p(N)}^p .
\]

We also need another inequality. For convenience, denote \( w_i = v_i - i d\mathcal{N} \), then

\[
\int_{B_{\varepsilon_0}^r} d\mathcal{H}^r(\xi) \int_M \frac{|d(w_i \circ \phi \circ u)|}{1 + |d(w_i \circ \phi \circ u)|} d\mathcal{H}^n(x) \leq \int_{B_{\varepsilon_0}^r} d\mathcal{H}^r(\xi) \int_M \frac{|d(w_i(u(x)))|}{1 + |d(w_i(u(x)))|} |du_x| d\mathcal{H}^n(x) \leq c(M,N) \int_M d\mathcal{H}^n(x) \int_N \frac{|dw_i(y)|}{1 + |dw_i(y)|} |du_x| d\mathcal{H}^m(y) \to 0
\]

as \( i \to \infty \) by Lebesgue's dominated convergence theorem. We use similar computation as in the proof of Lemma 7.1 for the last inequality. From (8.1), (8.2) and (8.3) we may find for each \( i \) a \( \xi_i \in B_{\varepsilon_0}^r \) such that

\[
\left\{ \begin{array}{l}
|d(v_i \circ \phi \circ u)|_{L^p(M)} \leq c(p,M,N) |du|_{L^p(M)} , \\
|v_i \circ \phi \circ u - \phi \circ u|_{L^p(M)} \leq c(p,M,N) |v_i - i d\mathcal{N}|_{L^p(N)} , \\
\int_M \frac{|d(v_i \circ \phi \circ u) - d(\phi \circ u)|}{1 + |d(v_i \circ \phi \circ u)|} d\mathcal{H}^n(x) \to 0 .
\end{array} \right.
\]
This clearly implies that after passing to a subsequence

\[
\left| d \left( \phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ u \right) \right|_{L^p(M)} \leq c(p, M, N) |du|_{L^p(M)},
\]

(8.5)

\[
\phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ u \to u \text{ in } L^p(M),
\]

\[
d \left( \phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ u \right) \to du \text{ a.e. on } M.
\]

Hence \( \phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ u \to u \) in \( W^{1,p}(M, N) \). Since \( \phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ u \in H^{1,p}_X(M, N) \), Proposition 8.3 follows.

Now we want to show in fact the proof above tells us more than Proposition 8.3. To state the result we need the following

**Definition 8.4** Let \( X \) be a path connected space which can be endowed with some CW complex structures, \( k \in \mathbb{Z}, k \geq 1 \). If we may find a continuous map \( f \) from \( X \) to itself such that \( f \) is \( k - 1 \) homotopic to \( id_X \) and \( f_*(\pi_k(X)) = 0 \), then we say \( X \) satisfies the \( k \)-vanishing condition.

We have the following basic facts about \( k \)-vanishing condition. Let \( X \) and \( Y \) be path connected spaces which can possess some CW complex structures, \( k \) be a natural number.

1. If either \( \pi_k(X) = 0 \) or \( X \) is \( k - 1 \) connected, then \( X \) satisfies the \( k \)-vanishing condition.

2. \( X \times Y \) satisfies the \( k \)-vanishing condition if and only if both \( X \) and \( Y \) satisfy the \( k \)-vanishing condition.

3. If \( Y \) satisfies the \( k \)-vanishing condition and there exists continuous maps \( f : X \to Y \) and \( g : Y \to X \) such that \( g \circ f \sim_{k-1} id_X \), then \( X \) satisfies the \( k \)-vanishing condition too.

4. \( \mathbb{R}^2 \) does not satisfy the 2-vanishing condition.

Now we may state the following

**Theorem 8.5** If \( 2 \leq p < n, p \in \mathbb{Z} \) and \( N \) satisfies the \( p \)-vanishing condition, then \( H^{1,p}_W(M, N) = \mathcal{H}^{1,p}_W(M, N) \) (the smallest subset of \( W^{1,p}(M, N) \) which contains \( C^\infty(M, N) \) and is closed under weak sequential convergence, see the paragraph after the proof of Theorem 5.5). Indeed for any \( u \in H^{1,p}_W(M, N) \), we may find a sequence \( u_i \in C^\infty(M, N) \) such that \( u_i \to u \) in \( W^{1,p}(M, N) \), \( |du_i|_{L^p(M)} \leq c(p, M, N) |du|_{L^p(M)} \) and \( du_i \to du \) a.e. on \( M \).
Proof: We only need to show \( \text{H}^1_w(M, N) \subset \text{H}^1_w(M, N) \). In the case \( p \geq \text{dim} N + 1 \), clearly we have \( \pi_p(N) = 0 \). It follows from the proof of Theorem 7.2 in [10] that \( \text{H}^1_w(M, N) \subset \text{H}^{2, p}_w(M, N) \subset \text{H}^1_w(M, N) \). Hence we assume \( p \leq \text{dim} N \). In this case it follows from Theorem 5.5 and Theorem 6.1 that we may find a sequence \( v_i \in C^\infty(N, N) \) such that \( v_i \to \text{id}_N \) in \( W^{1, p}(N, N) \), \( v_i \to \text{id}_N \) a.e. and \( v_i \to f \). In view of Theorem 4.1 and Theorem 6.1 in [10] we only need to show for any \( u \in \text{H}^1_w(M, N) \cap R^{p\infty}(M, N) \), we may find a sequence \( u_i \in C^\infty(M, N) \) such that \( u_i \to u \) in \( W^{1, p}(M, N) \), \( u_i \to u \) a.e. on \( M \) and \( |du_i|_{L^p} \leq c(p, M, N) |du|_{L^p} \). Under the same notations and arguments as in Proposition 8.3, we know (8.1), (8.2) and (8.3) are still true. Hence for each \( i \) we may find \( \xi_i \in B^\mathbb{R}_{\mathbb{C}} \) such that (8.4) is true. This implies that after passing to a subsequence (8.5) is satisfied. On the other hand, for the Lipschitz triangulation of \( M \), namely \( h : K \to M \) and the dual \( n - p - 1 \) skeleton \( L_{n-p-1} \) such that \( u \) is smooth on \( M \setminus h(L_{n-p-1}) \), if we denote \( M^i = h([K^i]) \), then \( u|_{M^{i+p-1}} \) has a continuous extension to \( M \), say \( \bar{u} \in C(M, N) \). For any \( p \) cell \( \Delta \), since \( u|_{\partial \Delta} = \bar{u}|_{\partial \Delta} \), it follows from \( (\phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i})_\ast (\pi_p(N)) = 0 \) that
\[
(\phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ u)|_\Delta \sim (\phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ \bar{u})|_\Delta
\]
hence
\[
(\phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ u)|_{M^p} \sim (\phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ \bar{u})|_{M^p}.
\]
By the homotopy extension theorem, we know \( (\phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ u)|_{M^p} \) has a continuous extension to \( M \). It follows from Theorem 6.2 in [10] that \( (\phi_{\xi_i}^{-1} \circ v_i \circ \phi_{\xi_i} \circ u)|_{M^p} \) is in \( \text{H}^2_w(M, N) \). Theorem 8.5 follows.

Corollary 8.6 If \( 2 \leq p < n, \ p \in \mathbb{Z}, \) and \( N \) satisfies the \( p \)-vanishing condition, then \( \text{H}^1_w(B_1^n, N) = W^{1, p}(B_1^n, N) \). In addition, for any \( u \in W^{1, p}(B_1^n, N) \), we may find a sequence \( u_i \in C^\infty(B_1^n, N) \) such that \( u_i \to u \) in \( W^{1, p}(B_1^n, N) \), \( |du_i|_{L^p(B_1^n)} \leq c(n, p, N) |du|_{L^p(B_1^n)} \) and \( du_i \to du \) a.e. on \( B_1 \).

Proof: Using Theorem 8.5 we see \( \text{H}^1_w(S^n, N) = \text{H}^1_w(S^n, N) \). On the other hand \( \text{H}^1_w(S^n, N) = W^{1, p}(S^n, N) \) because \( S^n \) is \( p-1 \) connected. Corollary 8.6 follows by considering \( S^n \) as the union of two disks.
9 A variational energy

Let $M$ be a smooth Riemannian manifold without boundary, $\Omega \subset M$ be a domain with compact closure and Lipschitz boundary. For $1 \leq p < \infty$, it is easy to prove that $C^\infty \left( \overline{\Omega}, N \right)$ is strongly dense in $L^p \left( \Omega, N \right)$. Let $u$ be a map in $W^{1,p} \left( \Omega, N \right)$.

Given $\varepsilon > 0$, if $u$ is a constant map, then we set $I_\varepsilon \left( u, p, \Omega, N \right) = 1$, otherwise, we define

$$I_\varepsilon \left( u, p, \Omega, N \right) = \inf \left\{ \frac{|dv|_{L^p(\Omega)}}{|du|_{L^p(\Omega)}} : v \in C^\infty \left( \overline{\Omega}, N \right) , |v - u|_{L^p(\Omega)} < \varepsilon \right\}.$$ 

It is clear that for $0 < \varepsilon_1 < \varepsilon_2$, we have $I_{\varepsilon_2} \left( u, p, \Omega, N \right) \leq I_{\varepsilon_1} \left( u, p, \Omega, N \right)$. Then we define

$$I \left( u, p, \Omega, N \right) = \lim_{\varepsilon \to 0^+} I_\varepsilon \left( u, p, \Omega, N \right) = \sup_{\varepsilon > 0} I_\varepsilon \left( u, p, \Omega, N \right).$$

For convenience, we write $I_\varepsilon \left( u, p, \Omega \right)$, $I_\varepsilon \left( u, \Omega \right)$ or $I_\varepsilon \left( u \right)$ when no confusion would happen. Similar convention applies for $I \left( u, p, \Omega, N \right)$.

**Lemma 9.1** $I \left( u \right) \geq 1$.

**Proof:** If for some $u$, we have $I \left( u \right) < \sigma < 1$, then $u$ is not a constant map and for each $k \in \mathbb{N}$, we may find a $u_k \in C^\infty \left( \overline{\Omega}, N \right)$ such that $|u_k - u|_{L^p(\Omega)} < \frac{1}{k}$ and $\frac{|du_k|_{L^p(\Omega)}}{|du|_{L^p(\Omega)}} < \sigma$. It follows from integration by parts formula that

$$|du|_{L^p(\Omega)} \leq \lim_{k \to \infty} \inf |du_k|_{L^p(\Omega)}.$$ 

Hence $|du|_{L^p(\Omega)} \leq \sigma |du|_{L^p(\Omega)}$, which implies $|du|_{L^p(\Omega)} = 0$, a contradiction. 

**Lemma 9.2** If $u_k \to u$ in $W^{1,p} \left( \Omega, N \right)$, then $I \left( u \right) \leq \liminf_{k \to \infty} I \left( u_k \right)$.

**Proof:** In view of Lemma 9.1, we may assume $\liminf_{k \to \infty} I \left( u_k \right) < \infty$ and $u$ is not a constant map. Given any $\varepsilon > 0$, we will show $I_\varepsilon \left( u \right) \leq \liminf_{k \to \infty} I \left( u_k \right)$. This clearly implies Lemma 9.2.

For any number $\lambda > \liminf_{k \to \infty} I \left( u_k \right)$, we may find infinitely many natural numbers $k$ such that $I \left( u_k \right) < \lambda$. For such $k$, we have $I_{\varepsilon_k} \left( u \right) \leq I \left( u_k \right) < \lambda$ and hence we may find a $v_k \in C^\infty \left( \overline{\Omega}, N \right)$ with $|v_k - u_k|_{L^p(\Omega)} < \varepsilon_k$. 


and \( \frac{|du_k|_{L^p(\Omega)}}{|du|_{L^p(\Omega)}} < \lambda \). For \( k \) large enough, we have \( |u_k - u|_{L^p(\Omega)} < \varepsilon \). This implies \( |v_k - u|_{L^p(\Omega)} < \varepsilon \) and

\[
I_\varepsilon(u) \leq \lambda \frac{|du_k|_{L^p(\Omega)}}{|du|_{L^p(\Omega)}}.
\]

Letting \( k \to \infty \), we get \( I_\varepsilon(u) \leq \lambda \).

The same proof as above gives us the following

**Lemma 9.3** If \( 1 < p < \infty \), \( u_k \rightharpoonup u \) in \( W^{1,p}(\Omega,N) \) and \( u \) is not a constant map, then

\[
I(u) \leq \liminf_{k \to \infty} I(u_k) \cdot \limsup_{k \to \infty} \frac{|du_k|_{L^p(\Omega)}}{|du|_{L^p(\Omega)}}.
\]

We remark that the conclusion of Lemma 9.3 remains true if we have \( p = 1 \), \( u_k, u \in W^{1,1}(\Omega,N) \), \( u_k \to u \) in \( L^1(\Omega) \) and \( \sup_{k \in \mathbb{N}} |du_k|_{L^1(\Omega)} < \infty \). The same proof goes through.

**Lemma 9.4** Assume \( 1 < p < \infty \), then \( u \in H_{S}^{1,p}(\Omega,N) \) if and only if \( I(u) = 1 \).

**Proof:** It is clear that for any \( v \in C^\infty(\overline{\Omega},N) \), \( I(v) \leq 1 \). This combines with Lemma 9.2 imply that for any \( u \in H_{S}^{1,p}(\Omega,N) \), \( I(u) \leq 1 \). Hence \( I(u) = 1 \) in view of Lemma 9.1.

On the other hand, assume \( I(u) = 1 \). If \( u \) is a constant map, then clearly \( u \in H_{S}^{1,p}(\Omega,N) \). Otherwise, for each \( k \in \mathbb{N} \), we may find a \( u_k \in C^\infty(\overline{\Omega},N) \) with \( |u_k - u|_{L^p(\Omega)} < \frac{1}{k} \) and \( |du_k|_{L^p(\Omega)} \leq \left(1 + \frac{1}{k}\right)|du|_{L^p(\Omega)} \). This clearly implies \( du_k \rightharpoonup du \) in \( L^p(\Omega) \). On the other hand, in view of the fact

\[
|du|_{L^p(\Omega)} \leq \liminf_{k \to \infty} |du_k|_{L^p(\Omega)} \leq \limsup_{k \to \infty} |du_k|_{L^p(\Omega)} \leq |du|_{L^p(\Omega)},
\]

we know \( |du_k|_{L^p(\Omega)} \to |du|_{L^p(\Omega)} \), and hence \( du_k \to du \) in \( L^p(\Omega) \). This implies \( u \in H_{S}^{1,p}(\Omega,N) \).

**Lemma 9.5** Assume \( 1 < p < \infty \), then \( u \in H_{W}^{1,p}(\Omega,N) \) if and only if \( I(u) < \infty \). Moreover, if \( u \in W^{1,p}(\Omega,N) \) is not a constant map, then

\[
I(u) = \inf \left\{ \liminf_{k \to \infty} \frac{|du_k|_{L^p(\Omega)}}{|du|_{L^p(\Omega)}} : u_k \in C^\infty(\overline{\Omega},N), u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega) \right\}.
\]

Here we use the convention that \( \inf \emptyset = \infty \).
PROOF: If $u \in H^{1,p}_W(\Omega, N)$ and it is not a constant map, then there exists a sequence $u_k \in C^\infty\left(\overline{\Omega}, N\right)$ such that $u_k \rightarrow u$ in $W^{1,p}(\Omega, N)$. It follows from Lemma 9.3 and Lemma 9.4 that

$$I(u) \leq \limsup_{k \rightarrow \infty} \frac{|du_k|_{L^p(\Omega)}}{|du|_{L^p(\Omega)}}.$$ 

By taking a subsequence, above inequality would imply that

$$I(u) \leq \liminf_{k \rightarrow \infty} \frac{|du_k|_{L^p(\Omega)}}{|du|_{L^p(\Omega)}}.$$ 

On the other hand, if $I(u) < \infty$ and $u$ is not a constant map, then for each $k \in \mathbb{N}$, we may find a $u_k \in C^\infty\left(\overline{\Omega}, N\right)$ such that $|u_k - u|_{L^p(\Omega)} < \frac{1}{k}$ and

$$|du_k|_{L^p(\Omega)} \leq \left(I(u) + \frac{1}{k}\right)|du|_{L^p(\Omega)}.$$ 

This clearly implies

$$\limsup_{k \rightarrow \infty} \frac{|du_k|_{L^p(\Omega)}}{|du|_{L^p(\Omega)}} \leq I(u)$$

and $u_k \rightarrow u$ in view of the fact $1 < p < \infty$. Hence $u \in H^{1,p}_W(\Omega, N)$.

The last identity in Lemma 9.5 follows from the argument above. Note that the definition of $I(u)$ for mappings resembles the classical definition of the Lebesgue area for sets.

Now we may state the main result of this section, which is sort a nonlinear version of Banach-Steinhaus theorem in linear functional analysis.

**Theorem 9.6** Given $1 \leq p < n$, if $H^{1,p}_W(B^1_1, N) = W^{1,p}(B^1_1, N)$, then we may find a positive constant $c(n, p, N)$ such that $I(u) \leq c(n, p, N)$ for any $u \in W^{1,p}(B_1, N)$.

To prove this theorem, we need some preparations.

**Lemma 9.7** Assume $1 \leq p < n$, $u \in W^{1,p}(B^1_1, N)$ such that $u|_{\partial B_1} \equiv y$, a point in $N$, $0 < \delta < \frac{1}{1}$. Pick up a maximal subset of $B_{1-2\delta}$, namely $x_1, \ldots, x_m$ such that $|x_i - x_j| \geq 3\delta$ for $i \neq j$. Define $u^i(x) = \begin{cases} u \left(\frac{x - x_i}{\delta}\right), & \text{if } x \in B_\delta(x_i), \\ y & \text{otherwise.} \end{cases}$
Then \( u^\delta \in W^{1,p}(B_1, N) \), \( u^\delta \big|_{\partial B_1} \equiv y \), \( \left| du^\delta \right|_{L^p(B_1)} \geq c(n, p) \delta^{-1} |du|_{L^p(B_1)} \)
and \( I \left( u^\delta \right) \geq I(u) \).

**Proof:** At first by the maximality of \( x_1, \ldots, x_m \), we have \( B_{1-2\delta} \subset \bigcup_{i=1}^m B_{2\delta}(x_i) \). Taking volume on both sides, we get
\[
\omega_n (1 - 2\delta)^n \leq m_\delta \omega_n (3\delta)^n,
\]
here \( \omega_n \) is the volume of \( B_1^n \). This implies \( m_\delta \geq c(n) \delta^{-n} \). An easy computation shows
\[
\left| du^\delta \right|_{L^p(B_1)}^p = m_\delta \delta^{n-p} |du|_{L^p(B_1)}^p \geq c(n) \delta^{-p} |du|_{L^p(B_1)}^p,
\]
and hence the third assertion is true. To prove \( I \left( u^\delta \right) \geq I(u) \), we may assume \( u \) is not a constant map. For any \( \varepsilon > 0 \), any \( v \in C^\infty \left( \overline{B_1}, N \right) \) with \( |v - u|_{L^p(B_1)} < \varepsilon \), it is clear that \( |v - u^\delta|_{L^p(B_\delta(x_1))} < \varepsilon \), hence we have
\[
\frac{\left| dv \right|_{L^p(B_1)}^p}{\left| du^\delta \right|_{L^p(B_1)}^p} \geq \frac{\sum_{i=1}^{m_\delta} \left| dv \right|_{L^p(B\delta(x_i))}^p}{m_\delta \delta^{n-p} |du|_{L^p(B_1)}^p} \geq \frac{\sum_{i=1}^{m_\delta} I_{\varepsilon} \left( u^\delta, B\delta(x_i) \right)^p}{m_\delta \delta^{n-p} |du|_{L^p(B_1)}^p} = I_{\varepsilon} \left( u^\delta, B\delta(x_1) \right)^p.
\]
This implies \( I_{\varepsilon} \left( u^\delta \right) \geq I_{\varepsilon} \left( u^\delta, B\delta(x_1) \right) \). Letting \( \varepsilon \to 0^+ \), we get
\[
I \left( u^\delta \right) \geq I \left( u^\delta, B\delta(x_1) \right) = I(u).
\]

**Lemma 9.8** If \( 1 \leq p < n \), then for any \( u \in W^{1,p}(B_1^n, N) \), we may find a \( \overline{u} \in W^{1,p}(B_1^n, N) \) such that \( \overline{u}(x) = u(2x) \) for \( x \in B_1^n \), \( \overline{u}|_{\partial B_1} \equiv \text{const} \) and \( \left| d\overline{u} \right|_{L^p(B_1)} \leq c(n, p) |du|_{L^p(B_1)} \).

**Proof:** At first we claim there exists a \( v \in W^{1,p}(B_1, N) \) such that \( v|_{B_{1/5}} \equiv \text{const} \), \( v|_{B_1 \setminus B_{1/5}} = u|_{B_1 \setminus B_{1/5}} \) and \( |dv|_{L^p(B_1)} \leq c(n, p) |du|_{L^p(B_1)} \).
To achieve this, we use the trick of “opening of maps” from [3]. We sketch the construction for reader’s convenience. An easy computation shows

$$\int_{B_{1/20}} d\xi \int_{B_1} \frac{|du (x)|^p}{|x - \xi|^n} dx \leq c(n) |du|^p_{L^p(B_1)}.$$ 

Hence we may find a $\xi \in B_{1/20}$ such that

$$\int_{B_1} \frac{|du (x)|^p}{|x - \xi|^n} dx \leq c(n) |du|^p_{L^p(B_1)}.$$ 

It follows from Poincare inequality and some computations that $\xi$ is a Lesbegue point of $u$, that is for some $y \in \mathbb{R}^d$,

$$\lim_{r \to 0^+} \int_{B_r(\xi)} |u (x) - y| dx = 0.$$ 

We may put $u (\xi) = y$. Define

$$v (x) = \begin{cases} 
  u (\xi), & x \in B_{13/20} (\xi), \\
  u \left( \xi + 14 \left( |x - \xi| - \frac{13}{20} \right) \frac{x - \xi}{|x - \xi|} \right), & x \in B_{14/20} (\xi) \setminus B_{13/20} (\xi), \\
  u (x), & x \in B_1 \setminus B_{14/20} (\xi). 
\end{cases}$$

Then $v$ satisfies all the requirements. Now we just set

$$\varpi (x) = \begin{cases} 
  u (2x), & x \in B_{1/2}, \\
  v \left( \frac{x}{2|x|^2} \right), & x \in B_1 \setminus B_{1/2}. 
\end{cases}$$

Clearly, $\varpi$ satisfies all the needed properties. 

**Proof of Theorem 9.6**: We will break the argument into several steps. Fix a point $y_0 \in N$.

**Step 1**: We claim that there exists a positive constant $c(n, p, N)$ such that for any $u \in W^{1, p} (B_1, N)$ with $u|_{\partial B_1} \equiv y_0$, we have $I (u) \leq c(n, p, N)$.

Indeed, if the above assertion is not true, then we may find a sequence $u_k \in W^{1, p} (B_1, N)$ such that $u_k|_{\partial B_1} \equiv y_0$ and $I (u_k) > \frac{4n}{2n - p}$. From Lemma 9.7 we know we may assume in addition that $|du_k|_{L^p(B_1)} \geq 1$, because otherwise we may replace $u_k$ by $u_k^{\delta_k}$ for some tiny positive $\delta_k$. Let $e_1 = (1, 0, \ldots, 0)$ be the first direction in $\mathbb{R}^n$. For convenience we set

$$\alpha_k = \frac{1}{2^{n-1} |du_k|_{L^p(B_1)}^{\frac{n-1}{n}}}.$$
Then define
\[
  u(x) = \begin{cases} 
    u_k \left( \frac{x - 2^{-k} e_1}{\alpha_k} \right), & \text{if } x \in B_{\alpha_k} \left( 2^{-k} e_1 \right), \\
    y_0, & \text{otherwise}.
  \end{cases}
\]

Computation shows
\[
  |du|^p_{L^p(B_1)} = \sum_{k=1}^{\infty} 2^{-4nk} = \frac{1}{2^{4n} - 1} < \infty.
\]

Hence \( u \in W^{1,p}(B_1, N) \). By assumption of Theorem 9.6, we may find a sequence \( v_i \in C^\infty \left( \overline{B_1}, N \right) \) such that \( v_i \rightharpoonup u \) in \( W^{1,p}(B_1, N) \). Fix a \( m \in \mathbb{N} \) and an \( \varepsilon > 0 \), we know for \( j \) large enough, \( |v_j - u|_{L^p(B_1)} < \varepsilon \), hence \( |v_j - u|_{L^p(B_{\alpha_k}(2^{-k} e_1))} < \varepsilon \) for any \( k \in \mathbb{N} \). For such \( j \), we have
\[
  |dv_j|^p_{L^p(B_1)} \geq \sum_{k=1}^{m} |dv_j|^p_{L^p(B_{\alpha_k}(2^{-k} e_1))} \geq \sum_{k=1}^{m} I_\varepsilon \left( u, B_{\alpha_k}(2^{-k} e_1) \right)^p |du|^p_{L^p(B_{\alpha_k}(2^{-k} e_1))} \geq \sum_{k=1}^{m} 2^{-4nk} I_\varepsilon \left( u, B_{\alpha_k}(2^{-k} e_1) \right)^p.
\]

Hence
\[
  \liminf_{j \to \infty} |dv_j|^p_{L^p(B_1)} \geq \sum_{k=1}^{m} 2^{-4nk} I_\varepsilon \left( u, B_{\alpha_k}(2^{-k} e_1) \right)^p.
\]

Letting \( \varepsilon \to 0^+ \), we get
\[
  \liminf_{j \to \infty} |dv_j|^p_{L^p(B_1)} \geq \sum_{k=1}^{m} 2^{-4nk} I \left( u, B_{\alpha_k}(2^{-k} e_1) \right)^p = \sum_{k=1}^{m} 2^{-4nk} \cdot 2^{4nk} = m.
\]

Letting \( m \to \infty \), we get \( \liminf_{j \to \infty} |dv_j|^p_{L^p(B_1)} = \infty \), which contradicts the fact that \( v_j \rightharpoonup u \) in \( W^{1,p}(B_1, N) \).

**Step 2**: We claim that there exists a positive constant \( c(n, p, N) \), such that for any \( u \in W^{1,p}(B_1, N) \) with \( u|_{\partial B_1} \equiv \text{const} \), we have \( I(u) \leq c(n, p, N) \).
Assume $u$ is not a constant map and $u|_{\partial B_1} = y$. We may find a smooth path $\phi : [0, 1] \to N$ such that $\phi(0) = y$, $\phi(1) = y_0$. For $\delta \in (0, 1/4)$, we have the map $u_\delta$ defined in Lemma 9.7. Let

$$u_\delta(x) = \begin{cases} u_\delta(2x), & \text{if } x \in B_{1/2}, \\ \phi(2|x| - 1), & \text{otherwise.} \end{cases}$$

From step 1 we know $I(v) \leq c(n, p, N)$. On the other hand, for any $\varepsilon > 0$ and $v \in C^\infty(\overline{B_1}, N)$ with $|v - u_\delta|_{L^p(B_1)} < \varepsilon$, we have $|v - u_\delta|_{L^p(B_{1/2})} < \varepsilon$, hence

$$\frac{|dv|_{L^p(B_1)}}{|du_\delta|_{L^p(B_1)}} \geq \frac{|dv|_{L^p(B_{1/2})}}{|du_\delta|_{L^p(B_{1/2})}} \cdot \frac{|du_\delta|_{L^p(B_{1/2})}}{|du_\delta|_{L^p(B_1)}} \geq I_\varepsilon(u_\delta, B_{1/2}) \cdot \frac{|du_\delta|_{L^p(B_{1/2})}}{|du_\delta|_{L^p(B_1)}}.$$  

This implies

$$I_\varepsilon(u_\delta) \geq I_\varepsilon(u_\delta, B_{1/2}) \cdot \frac{|du_\delta|_{L^p(B_{1/2})}}{|du_\delta|_{L^p(B_1)}}.$$  

Letting $\varepsilon \to 0^+$, we get

$$c(n, p, N) \geq I(u) \geq I\left(u_\delta, B_{1/2}\right) \cdot \frac{|du_\delta|_{L^p(B_{1/2})}}{|du_\delta|_{L^p(B_1)}} \geq I\left(u_\delta, B_{1/2}\right) \cdot \frac{|du_\delta|_{L^p(B_{1/2})}}{|du_\delta|_{L^p(B_1)}}.$$  

Letting $\delta \to 0^+$, in view of Lemma 9.7, we get $I(u) \leq c(n, p, N)$. This finishes step 2.

Now we will prove Theorem 9.6 in its full generality. Given any $u \in W^{1, p}(B_1, N)$, we may find a corresponding map $\pi$ as in Lemma 9.8. From step 2 we know $I(\pi) \leq c(n, p, N)$. On the other hand, the same proof of step 2 shows

$$I(\pi) \geq I(\pi, B_{1/2}) \cdot \frac{|d\pi|_{L^p(B_{1/2})}}{|d\pi|_{L^p(B_1)}} = I(u) \frac{|d\pi|_{L^p(B_{1/2})}}{|d\pi|_{L^p(B_1)}}.$$  

Hence

$$I(u) \leq c(n, p, N) \frac{|d\pi|_{L^p(B_1)}}{|d\pi|_{L^p(B_{1/2})}} = c(n, p, N) \frac{|d\pi|_{L^p(B_{1/2})}}{|du|_{L^p(B_1)}} \leq c(n, p, N).$$
10 Defect measures and minimal connections

Let $p \geq 2$, $p \in \mathbb{Z}$, and let $u \in H^1_{W}(M, N)$. Then by Lemma 9.5, one has $I(u) < \infty$. In particular, there exists a sequence $u_k \in C^\infty(M, N)$ such that $u_k \rightharpoonup u$ in $W^1_p(M, N)$ with

$$\int_M |du_k|^p \ dx \leq (I(u) + \varepsilon)^p \int_M |du|^p \ dx$$

for every $\varepsilon > 0$ and for all $k \geq k(\varepsilon)$. If one considers a sequence of Radon measures $\mu_k$,

$$\mu_k = |du_k|^p \ dx \rightharpoonup \mu = |du|^p \ dx + \nu.$$

Here $\nu \geq 0$ is also a Radon measure. It is natural to study the structure of the defect measure $\nu$. From the discussion in [13], one believes it should be a generalization of “minimal connections” studied in [2].

For this purpose, we first consider a map $u$ satisfies the assumption of Theorem 5.5. That is for a Lipschitz rectilinear cell decomposition $h : K \to M$, let $M^i = h([K^i])$ for $i \geq 0$, and $L^{n-p-1}$ be a dual skeleton, $u \in W^1_p(M, N)$ such that it is continuous on $M \setminus h(L^{n-p-1})$. For $p \geq 2$, a positive integer, Theorem 5.5 says $u \in H^1_{W}(M, N)$ if and only if $u|_{M^{p-1}}$ has a continuous extension to $M$. Assume this topological condition is satisfied, let $\{u_k\}$ be as above such that

$$\int_M |du_k|^p \ dx \to \int_M |du|^p \ dx + \nu(M) = I(u)^p \int_M |du|^p \ dx.$$

THEOREM 10.1 Let $u \in H^1_{W}(M, N)$ satisfy the assumption of Theorem 5.5. Then $\nu = \Theta(x)\mathcal{H}^{n-p}[\Sigma]$, where $0 < \varepsilon_0(p, M, N) \leq \Theta(x) \leq C(u, M, N)$, $\Sigma$ is a closed $n-p$ rectifiable subset of $M$. Moreover for $\mathcal{H}^{n-p}$ a.e. $x$,

$$\Theta(x) = \inf \left\{ E_p(v) : v \in W^1_p(S^p, N), [v] = \alpha_x \right\},$$

for some nontrivial class $\alpha_x \in \pi_p(N)$, where $E_p(v) = \int_{S^p} |dv|^p$.

From the work of Duzaar-Kuwert [5], we see that

$$\Theta(x) = \sum_{j=1}^{l} E_p(\phi_j).$$
Here each $\phi_j : S^p \to N$ is a $C^{1,\alpha}$, $p$-harmonic map which is an energy minimizing map from $S^p$ into $N$ within the topological class $[\phi_j]$. Moreover, these $\phi_j$‘s are a decomposition of $\alpha_x \in \pi_p(N)$ (see [5] for the details).

As in Section 5 of [13], it is easy to see that $\Theta(x)$ takes discrete values. Therefore $\mathbb{V} = V(\Theta, \Sigma)$ can be viewed as an integral varifold. We shall study the structure of defect measures for general $u \in H^1_{W^p}(M, N)$, which may not satisfy the additional assumption as in Theorem 5.5, elsewhere. It can be deduced from modifications of arguments in [13]. Here we shall simply point out that away from the singularity of $u$ (which is in fact lower dimensional), the defect measure $\nu$ is as described in Theorem 10.1. For simplicity, we shall assume that the map $u$ is smooth away from the singularities.

Indeed, let $B_{\rho_0} \subset M$ be a ball such that $u \in C^1(\overline{B_{\rho_0}}, N)$ and osc$_{B_{\rho_0}} u \leq \varepsilon_1 \ll 1$. Let $\{u_k\}$ be a sequence such that (10.1) is valid. We consider $u_k|_{B_{\rho_0}}$. For any $B_{\rho} \subset B_{\rho_0}$, we consider $\{\bar{u}_k\} \subset C^1(\overline{B_{\rho}}, N)$ such that $\bar{u}_k = u_k$ on $\partial B_{\rho}$ and $\{\bar{u}_k\}$ is a $p$-energy minimizing sequence in $C^1(\overline{B_{\rho}}, N)$ subject to the Dirichelet boundary condition. Since $u_k \to u$ on $B_{\rho}$, we conclude $\bar{u}_k \to \bar{a}$. Here $\bar{a}$ is the $p$-energy minimizing map in $B_{\rho}$ with $\bar{a}|_{\partial B_{\rho}} = u|_{\partial B_{\rho}}$. We let

$$v_k(x) = \pi_N (\bar{a}_k(x) + (u(x) - \bar{a}(x))) \quad \text{for } x \in B_\rho.$$

Since $|\bar{a} - u|_{L^\infty(B_{\rho})} \leq c\varepsilon_1 \ll 1$, it is easy to see $v_k \to u$ on $B_{\rho}$. Moreover, $v_k \in C(\overline{B_\rho}, N)$ and $v_k = \bar{a}_k = u_k$ on $\partial B_{\rho}$. A simple estimate yields

$$\int_{B_\rho} |du|^p \, dx + \nu(B_\rho) \leq \liminf_k \int_{B_\rho} |dv_k|^p \, dx = \int_{B_\rho} |du|^p \, dx + \nu(B_\rho)$$

$$\leq \liminf_k \int_{B_\rho} |d\bar{a}_k(x)|^p \, dx + c_0 \int_{B_\rho} |d(u - \bar{a})|^p \, dx.$$

Here we assume

$$|dv_k|^p \, dx \to |du|^p \, dx + \eta.$$

Let us assume

$$|d\bar{a}_k|^p \, dx \to |d\bar{a}|^p \, dx + \lambda.$$

Then the above estimate implies

$$\int_{B_\rho} |du|^p \, dx + \nu(B_\rho) \leq \int_{B_\rho} |d\bar{a}|^p \, dx + \lambda(B_\rho) + o(|B_\rho|).$$

On the other hand,

$$\int_{B_\rho} |d\bar{a}|^p \, dx + \lambda(B_\rho) \leq \int_{B_\rho} |du|^p \, dx + \nu(B_\rho),$$
by the trivial comparison. Therefore

|\lambda(B_\rho) - \nu(B_\rho)| \leq o(|B_\rho|).

Applying the results in [13], one has \( \lambda = \mathcal{T} \cdot \mathcal{H}^{n-p}[\Sigma] \) as described in the conclusion of Theorem 10.1, in \( B_{\rho_0} \), thus \( \nu \) (which is singular with respect to the Lebesgue measure) has to be also of the same form.

Now we want to point out an interesting case \( W^{1,3}(B^4, S^2) \) that is related to the recent work of Hardt-Rivière [12]. In this case, those maps from \( B^4 \) to \( S^2 \), smooth with exception of finite number of isolated singularities are strongly dense in \( W^{1,3}(B^4, S^2) \). Let \( u \) be such a map, i.e., \( u \in C^\infty(\overline{B^4 \setminus \{x_1, \ldots, x_k\}, S^2}) \) for \( x_1, \ldots, x_k \in B^4 \). Let \( r \) be a small and positive number, one has a well-defined integer \( d_j \), the Hopf degree of \( u|_{\partial B_r(x_j)} \). There is a sequence \( u_i \in C^\infty(\overline{B^4}, S^2) \) such that \( u_i \rightharpoonup u \) in \( W^{1,3}(B^4, S^2) \) and

\[
\int_{B^4} |du_i|^3 \to I(u)^3 \int_{B^4} |du|^3.
\]

From [12] one knows

\[
G_{u_i} \to G_u + I \times S^3
\]
as scans. On the other hand, from the discussion in Section 5 of [13] and Theorem 10.1, one has

\[
|du_i|^3 dx \to |du|^3 dx + \nu, \quad \nu = \Theta \cdot \mathcal{H}^{1}[\Sigma].
\]

\( \forall = V(\Theta, \Sigma) \) is an one dimensional integral varifold. If we define \( \mathcal{T} = \langle \overline{T}, \Theta, \Sigma \rangle \), here the orientation of \( \overline{T} \) is chosen according to whether the Hopf degree of \( \alpha_x \in \pi_3(S^2) \) is positive or negative. More precisely, we require

\[
\overline{T}(x) \wedge \widehat{e}_1(x) \wedge \widehat{e}_2(x) \wedge \widehat{e}_3(x) = \text{sign of the degree of } \alpha_x,
\]

here \( \widehat{e}_1(x) \wedge \widehat{e}_2(x) \wedge \widehat{e}_3(x) \) is a given orientation on the 3-plane orthogonal to \( \overline{T}(x) \) so that the map restricted to it gives the Hopf degree of \( \alpha_x \). Then \( \mathcal{T} \) is an integral rectifiable current of finite mass. We note that \( \Theta(x) \approx |\text{deg } \alpha_x|^2 \) by the earlier result of Rivière [17], hence the final conclusion is consistent with [12].

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