There is a natural way to define functions of a self-adjoint operator by using Corollary 77.31.1 Given a bounded Borel function h on $\mathbb{R}$ we define (in the notation of 77.31.1)

(77.32.1) $$h(A) = U^{-1} T_{h(F)} U$$

where $T_{h(F)}$ is the operator of multiplication by $h(F(m))$ on $L^2(M, d\mu)$. Using this definition the following Theorem follows easily from Corollary 77.31.1

Lecture 7 **Theorem 77.32.2** (spectral theorem - functional calculus form)

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Then there is a unique map $\hat{\phi}$ from the bounded Borel functions $\mathcal{B}$ on $\mathbb{R}$ into $\mathcal{L}(\mathcal{H})$ so that

(a) $\hat{\phi}$ is an algebraic $*$-homomorphism

(b) $\hat{\phi}$ is norm continuous, i.e., $\|\hat{\phi}(h)\|_{\mathcal{L}(\mathcal{H})} \le \|h\|_\infty$

(c) Let $h_n(x)$ be a sequence of bded Borel functions with $h_n(x) \to x$ as $n \to \infty$ for each x, and $|h_n(x)| \le |x|$ for all x and n. Then for any $\psi \in D(A)$

$$\lim_{n\to\infty} \hat{\Phi}(h_n)\psi = A\psi$$

(d) If $h_n(x) \to h(x)$ pointwise and if the sequence $\|h_n\|$ is bded, then

$$\hat{\Phi}(h_n) \to \hat{\Phi}(h)$$

strongly

(e) If $A\psi = \lambda\psi$, then $\hat{\Phi}(h)\psi = h(\lambda)\psi$

(f) If $h \ge 0$, then $\hat{\Phi}(h) \ge 0$.

Proof: Exercise. Hint: Show that if $h(x) = \frac{1}{x-z}$, $z \in \mathbb{C} \setminus \mathbb{R}$ then $h(A) = \frac{1}{A-z}$, the resolvent of A. $\square$

Amongst the most important functions $h(A)$ in the functional calculus for A, are the projection operators mentioned in our first lecture

$$P_\Omega = \chi_\Omega(A)$$

where Ω is a Borel subset of $\mathbb{R}$. As $h \to h(A)$ is a (*)-homomorphism

$$P_\Omega^2 = \chi_\Omega(A)\chi_\Omega(A) = \chi_\Omega^2(A) = \chi_\Omega(A) = P_\Omega$$

and

$$P_\Omega^\dagger = \bar{\chi}_\Omega(A) = \chi_\Omega(A) = P_\Omega$$

so P_Ω is a self-adjoint projection for each Ω. The fact that $h \to h(A)$ is a (*)-homomorphism implies that for any $\phi \in \mathcal{H}$, $(\phi, P_\Lambda \phi)$ is a well-defined Borel measure on $\mathbb{R}$, which we denote by $d(\phi, P_\lambda \phi)$ (note that $\chi_{\bigcup_j \Lambda_j} = \sum \chi_{\Lambda_j}$ if the Λ_j's are disjoint)

We have the decomposition

$$A = \int \lambda \, dP_\lambda$$

in the sense that for any $\phi \in D(A)$

(78.1) $$(\phi, A\phi) = \int \lambda \, d(\phi, P_\lambda \phi).$$

To verify (78.1), we note that for $\psi = U\phi$ (here we use U from Th^m 69.1)

$$(\phi, P_\Lambda \phi) = \int_\Lambda \sum_{n=1}^{N} |\psi(\lambda, n)|^2 \, d\mu_n(\lambda)$$

and hence $d(\phi, P_\lambda \phi)$ is just the measure $\sum_{n=1}^{N} |\psi(\lambda, n)|^2 d\mu_n(\lambda)$

Thus if $\phi \in D(A) = U^{-1}\left(\{\psi : \lambda \in L^2\left(\sum_{n=1}^{N} |\psi(\lambda,n)|^2 d\lambda\right) = L^2(d(\phi, P_\lambda \phi))\}\right)$

we have

$$(\phi, A\phi) = \int \lambda \sum_{n=1}^{N} |\psi(\lambda, n)|^2 d\mu_n(\lambda) = \int \lambda \, d(\phi, P_\lambda \phi).$$

In addition to (78.1), we clearly also have the following result. If $h(x)$ is a bded Borel function and $\phi \in \Phi$, then

(79.1) $$(\phi, h(A)\phi) = \int h(\lambda)\, d(\phi, E_\lambda \phi).$$

Lecture 7 Remark 80.1 (To be inserted right after Th^m 69.1 when we begin to discuss multiplicity).

We have seen, even in the matrix case, that there is a lot of freedom in choosing the measures $\{d\mu_n(\lambda)\}$. Indeed, if the $m\times m$ matrix $A = A^*$ has simple spectrum, $\lambda_1, \ldots, \lambda_m$, say, then

$$d\mu(\lambda) = \sum_{i=1}^{m} a_i\, \delta(\lambda - \lambda_i)$$

is an acceptible spectral measure for A as long as all $a_i > 0$. Said differently, as long as all the measures have the same support $\{\lambda_1, \ldots, \lambda_m\}$. More generally, one easily sees that if A is s.adj. and spectrally simple, i.e. only 1 meas. $d\mu(\lambda)$ is needed in the spec. th^m for A, then $d\mu(\lambda)$ can be replaced by any other measure $d\tilde{\mu}(\lambda)$ which is <u>mutually absolutely</u>

continuous with resp. to $d\mu(\lambda)$: or equivalently, by the Radon–Nikodym theorem, $d\mu(\lambda)$ and $d\tilde{\mu}(\lambda)$ have the same sets of measure zero. (why?). In the general case, one splits up $\mathbb{R}$ into a disjoint union of sets, $\mathbb{R} = \bigcup_{j=1}^{N} B_j$, $B_j \cap B_k = \phi$ if $j \neq k$, and for all $\lambda \in B_j$, A has fixed multiplicity j. One then shows that the measures $d\mu_j$ associated with B_j are determined up to any measure $d\tilde{\mu}_j$ with the same sets of measure zero (see RS Vol I, Th^m VII.6 for details). This is the sense in which the measures are uniquely determined.

Z

We now investigate the connection between spectral measure and spectrum.

The <u>support</u>, $\operatorname{supp}\mu$, of a measure on $\mathbb{R}$ is the complement of the largest open set on which μ vanishes. Such a largest open set exists, as we now show.

If $\mu(O) > 0$ for all open sets O, then clearly $\operatorname{supp}\mu = \mathbb{R}$. Suppose $\mu(O) = 0$ for some open set $O \neq \phi$, and let $B = \bigcup_\alpha O_\alpha$ be the union of all open sets O_α st $\mu(O_\alpha) = 0$. Clearly B is open. We show that $\mu(B) = 0$. This shows that ~~indeed~~ the largest open set on which μ vanishes indeed exists. But as B is open, $B = \bigcup_j I_j$, where the I_j's are open intervals in $\mathbb{R}$, $I_j = (a_j, b_j)$. Let $a_j < a_j' < b_j' < b_j$.

then $[a_i', b_i'] \subset \bigcup_\alpha O_\alpha$ and by the Heine-Borel Theorem, $[a_i', b_i'] \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_m}$ for some finite sub-collection of the O_α's. Thus

$$\mu([a_i', b_i']) \le \sum_k \mu(O_{\alpha_k}) = 0$$

Letting $a_i' \downarrow a_i$, $b_i' \uparrow b_i$ we see that $\mu(I_i) = 0$ and hence $\mu(B) = 0$.

It follows that

(83.1) $$\operatorname{supp}\mu = \{\lambda : \mu(\lambda-\varepsilon, \lambda+\varepsilon) > 0 \quad \forall \varepsilon > 0\}.$$

Indeed for B as above, $\operatorname{supp}\mu = B^c$. Suppose $\mu(\lambda-\varepsilon, \lambda+\varepsilon) > 0$ $\forall \varepsilon > 0$. If $\lambda \in B$, then $(\lambda-\varepsilon, \lambda+\varepsilon) \subset B$ for some $\varepsilon > 0$ as B is open. Hence $\mu(\lambda-\varepsilon, \lambda+\varepsilon) \le \mu(B) = 0$, which is a contradiction. Hence $\lambda \in B^c = \operatorname{supp}\mu$. Conversely, suppose $\lambda \in \operatorname{supp}\mu = B^c$. If $\mu(\lambda-\varepsilon, \lambda+\varepsilon) = 0$ for some $\varepsilon > 0$, then $(\lambda-\varepsilon, \lambda+\varepsilon) \subset B$, as B is the largest open set with this property. In particular $\lambda \in B$, which is a contradiction. Thus $\mu(\lambda-\varepsilon, \lambda+\varepsilon) > 0$ $\forall \varepsilon > 0$ and this proves (83.1).

<u>Definition</u>

Let $\{\mu_n\}_{n=1}^{N}$ be the family of measures assoc. with $A = A^*$. The <u>support</u> of the family $\{\mu_n\}_{n=1}^{N}$, denoted $\operatorname{supp}\{\mu_n\}_{n=1}^{N}$, is (again) <u>defined</u> as the complement of the largest open set B on which all the μ_n's vanish. The same argument as above shows that B indeed exists.

We have

$$\text{(84.1)} \qquad \operatorname{supp}\{\mu_n\}_{n=1}^{N} = \overline{\left(\bigcup_{n=1}^{N} \operatorname{supp}\mu_n\right)}$$

Indeed as $\mu_n(B) = 0$ $\forall n$ $\Rightarrow$ $B \subset (\operatorname{supp}\mu_n)^c$ (= largest open set on which μ_n vanishes). Hence $B \subset \bigcap_n (\operatorname{supp}\mu_n)^c$

$\Rightarrow B^c \supset \bigcup_n \operatorname{supp}\mu_n \Rightarrow B^c \supset \overline{\bigcup_n \operatorname{supp}\mu_n}$, as B^c is closed. Conversely, suppose $\mu_m\left(\left(\overline{\bigcup_{n=1}^N \operatorname{supp}\mu_n}\right)^c\right) > 0$ for some m. Then $\mu_m\left(\left(\bigcup_{n=1}^N \operatorname{supp}\mu_n\right)^c\right)$ $\geq \mu_m\left(\left(\overline{\left(\bigcup_{n=1}^N \operatorname{supp}\mu_n\right)}\right)^c\right) > 0$. But then, in particular $\mu_m((\operatorname{supp}\mu_m)^c) > 0$, which is a contradiction. Hence $\mu_m\left(\left(\overline{\bigcup_{n=1}^N \operatorname{supp}\mu_n}\right)^c\right) = 0 \quad \forall m$. But $\left(\overline{\bigcup_{n=1}^N \operatorname{supp}\mu_n}\right)^c$ is open and hence $\left(\overline{\bigcup_{n=1}^N \operatorname{supp}\mu_n}\right)^c \subset B$ i.e. $\overline{\bigcup_{n=1}^N \operatorname{supp}\mu_n} \supset B^c$. This proves (84.1).

The main result is the following

__Th^m 85.1__ Let $A = A^+$ with spec. measures $\{\mu_n\}_{n=1}^N$. Then

(85.2) $$\sigma(A) = \operatorname{supp}\{\mu_n\}_{n=1}^N$$

Before we prove this result, we consider the following.

__Definition:__ Let F be a real valued mble function on

We say λ is in the <u>essential range</u> of F iff

(86.1) $$\mu(m : \lambda - \varepsilon < F(m) < \lambda + \varepsilon) > 0 \qquad \forall \varepsilon > 0.$$

Thus if $M = \mathbb{R}$ and $F(\lambda) = \lambda$

(86.2) $$\text{ess range } F = \text{supp } \mu$$

<u>Proposition 86.3</u> Let F be a real-valued measurable function on a σ-finite measure space (M, μ) and suppose F is finite μ-a.e., be the operator of mult. by F on $L^2(M, d\mu)$,

$$(T_F\, g)(m) = F(m)\, g(m)$$

with domain $D(F) = \{ g \in L^2(d\mu) : Fg \in L^2(d\mu) \}$.

Then T_F is s.adj on $L^2(d\mu)$ and

$$\sigma(T_F) = \text{ess. ran. of } F.$$

<u>Proof:</u> The fact that T_F is densely defined and s.adj is a simple <u>exercise</u>

Suppose $\lambda \notin$ ess ran F. Then for some $\varepsilon > 0$,

(87.1) $$\mu\left(m: \quad F(m) \in (\lambda-\varepsilon, \lambda+\varepsilon)\right) = 0.$$

Let $$G_\lambda(m) = \frac{1}{F(m)-\lambda}\, \chi_{\{|F(m)-\lambda| > \frac{\varepsilon}{2}\}}.$$

Then T_{G_λ}, mult. by G_λ, is clearly a bded operator in $L^2(d\mu)$. If $g \in D(T_F)$, then

$$T_{G_\lambda}(T_F-\lambda) g(m) = \frac{1}{F(m)-\lambda}\, \chi_{|F-\lambda|> \frac{\varepsilon}{2}} (F(m)-\lambda) g(m)$$

$$= \chi_{(|F-\lambda| > \varepsilon/2)}(m)\; g(m)$$

But $\chi_{|F-\lambda| \leq \varepsilon/2}(m) = 0$ in $L^2(M, d\mu)$ by (87.1)

Hence $T_{G_\lambda}(T_F-\lambda)\, g(m) = g(m)$. ie

(87.2) $$T_{G_\lambda}(T_F-\lambda) = 1_{D(T_F)}$$

Similarly, as $G_\lambda : \mathcal{H} \to D(T_F)$

$$(T_F-\lambda)\, T_{G_\lambda} g(m) = g(m) \qquad \forall g \in L^2(d\mu).$$

ie

(87.3) $$(T_F-\lambda)\, T_{G_\lambda} = 1_{L^2(d\mu)}.$$

This shows that $\lambda \in \rho(T_F)$.

Now suppose that $\lambda \in$ ess. range of F. Then for any $\varepsilon > 0$, let $g_\varepsilon^{(j)} = \chi_{\{|F(m)-\lambda|<\varepsilon\}}(m)\,\chi_{A_j}(m)$ where $M = \bigcup_j A_j$, $\mu(A_j) < \infty$. Clearly, we can always assume that $A_j \uparrow$ ie $A_j \subset A_{j+1}$, $j \geq 1$. Now as $\mu(|F(m)-\lambda|<\varepsilon) > 0$, there $\exists\, j_0$ st $g_\varepsilon^{(j_0)} \neq 0$ and so $g_\varepsilon^{(j)}$ is a non-zero vector in $L^2(\mu)$ (here we use $\mu(A_j) < \infty$). As $|(F(m)-\lambda)\,g_\varepsilon^{(j_0)}| \leq \varepsilon\,\chi_{A_{j_0}}(m) \in L^2(d\mu)$, we see that $g_\varepsilon^{(j_0)} \in D(T_F)$. Hence

(88.1) $$\|(T_F-\lambda)\,g_\varepsilon^{(j_0)}\|^2_{L^2(d\mu)} = \int_{A_j} |(F(m)-\lambda)\,\chi_{|F-\lambda|<\varepsilon}(m)|^2\,d\mu$$

$$\leq \varepsilon^2 \int |g_\varepsilon^{(j_0)}|^2\,d\mu$$

$$= \varepsilon^2\,\|g_\varepsilon^{(j_0)}\|^2$$

On the other hand if $\lambda \in \rho(T_F)$ then $(T_F-\lambda)^{-1}$ is bdd, $\|(T_F-\lambda)^{-1}\| \leq c < \infty$ and so

(88.2) $$\|g_\varepsilon^{(j_0)}\|^2 = \|(T_F-\lambda)^{-1}(T_F-\lambda)\,g_\varepsilon^{j_0}\|^2 \leq c^2\,\|(T_F-\lambda)\,g_\varepsilon^{(j_0)}\|^2$$

Inserting (88.2) into (88.1) $\Rightarrow$

$$c^{-2} \| g_\varepsilon^{(j_0)} \|^2 \leq \varepsilon^2 \| g_\varepsilon^{(j_0)} \|^2$$

Choosing $\varepsilon < \frac{1}{c}$, we obtain a contradiction, as $g_\varepsilon^{(j_0)} \neq 0$. Thus $\lambda \in \sigma(T_F)$. This completes the proof of Prop. 86.3.

Corollary 89.1

If A is unitarily equivalent to mult. by x on $(\mathbb{R}, d\mu)$, $\mu(\mathbb{R}) < \infty$, then

$$\sigma(A) = \text{ess range of mult. by } x$$

$$= \{\lambda \in \mathbb{R} : \mu(\lambda - \varepsilon < x < \lambda + \varepsilon) > 0 \ \forall \varepsilon > 0\}$$

$$= \text{supp } \mu$$

We now prove Th^{m} 85.1

As A is unit. equiv to mult. by λ on $\bigoplus_{n=1}^{N} L^2(\mathbb{R}, d\mu_n)$ it is clear from the above that if $\lambda \in \text{supp } \mu_n$ for any n

then $\lambda \in \sigma(A)$. Thus $\bigcup_{n=1}^{N} \operatorname{supp} \mu_n \subset \sigma(A)$ and hence $\overline{\left(\bigcup_{n=1}^{N} \operatorname{supp} \mu_n\right)} \subset \sigma(A)$ as $\sigma(A)$ is closed.

On the other hand, if $\lambda_0 \notin \overline{\bigcup_{n=1}^{N} \operatorname{supp} \mu_n}$, then $\lambda_0 \in B$ = largest open set s.t. $\mu_n(B) = 0$ $\forall n$. It follows in particular that $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subset B$ for some $\varepsilon > 0$.

It is now easy to show as above (p87 et seq) that for

$$G_{\lambda_0}(\lambda) = \frac{1}{\lambda - \lambda_0} \chi_{|\lambda - \lambda_0| > \varepsilon/2}$$

the bounded operator $G_{\lambda_0}(A)$ provides an inverse for $A - \lambda_0$. Thus $\lambda_0 \in \rho(A)$. This proves Th^m 85.1.

Lecture 8 Exercise 90.1

$$\lambda \in \sigma(A) \iff P_{(\lambda-\varepsilon, \lambda+\varepsilon)} \neq 0 \quad \forall \varepsilon > 0.$$