Lecture 4 We learn the following from this exple:

(52.1) We see that T_3 differs from $T_3^* = T_1$ only in the choice of boundary conditions. This is a very general situation for differential operators i.e. they are both given by the same (formal) symbol: only when we add in boundary conditions (read "physical requirements") do we see the difference.

(52.2) $\sigma(T_3^*) = \sigma(T_1) = \mathbb{C}$ by (45.1).

and also

(52.3) $\sigma(T_3) = \mathbb{C}$

Indeed if $\lambda \in \mathbb{C}$, set $g_\lambda(t) = e^{-i\lambda t} \in L^2(0,1)$.

Then we cannot have $(\lambda - T_3) f = g_\lambda$ for any $f \in D(T_3)$. Indeed we would have $-i \frac{d}{dx}(e^{i\lambda x} f) = e^{i\lambda x} g_\lambda$ so that $-i\, f(x) e^{i\lambda x} = -i\, f(0) e^{i\lambda 0} + \int_0^x g_\lambda e^{i\lambda t}$

But as $f(0) = f(1) = 0$, we must have

$\int_0^1 g_\lambda e^{i\lambda t} dt = 0$. But $g_\lambda = e^{-i\lambda t}$ so then

$\int_0^1 g_\lambda e^{-i\lambda t} dt = 1$. Hence $(\lambda - T_3)$ is not onto.

However it is easy to see that $\lambda - T_3$ is 1-1.

On the other hand our previous calculations show that $\lambda - T_3^*$ is not 1-1, but it is easy to see that $\lambda - T_3^*$ is onto for any λ. Thus although $\sigma(T_3^*) = \mathbb{C} = \sigma(T_3)$, we see that the nature of the spectra are very different.

<u>Definition</u>

A symmetric op. T is called <u>essentially self-adjoint</u> (e.s.a.) if $\overline{T}$ is s.adj. If T is closed, $D \subset D(T)$ is called a <u>core</u> for T if $\overline{T \upharpoonright D} = T$

Now if T is e.s.a it has only 1 s.adj extension. Indeed if S is a s.adj. ext. of T, $T \subset S$, then as $S = S^* =$ closed, we must have $\bar{T} \subset S$. Hence $S = S^* \subset \bar{T}^* = \bar{T}$

Thus $S = \bar{T}$. The converse is also true i.e. if T has only 1 s.adj. extension, then necessarily T is e.sa. (see R Simon Vol 2 Section X.1 & also see later in these lectures)

The importance of e.s.a. is that one is often given a non-closed sym. op. T, say, to consider. If it can be shown to be e.s.a., then there is a unique way of assoc. to T a s.adj. operator by extension, viz, $\bar{T}$.

Now suppose that T is s.adj., $T = T^*$ and $\exists$

$\varphi \neq 0$, $\varphi \in D(T) = D(T^*)$ st $T^*\varphi = i\varphi$, Then

$T\varphi = i\varphi$ ~~and~~ $(T\varphi, \varphi) = (\varphi, T^*\varphi)$

$\Rightarrow \quad (i\varphi, \varphi) = (\varphi, i\varphi).$

$\Rightarrow \quad -i(\varphi, \varphi) = i(\varphi, \varphi)$

$\Rightarrow \quad \varphi = 0$

Similarly $T^*\varphi = -i\varphi = 0 \Rightarrow \varphi = 0$. It is an interesting and useful fact that the converse is true ie if T is a closed sym. op and $T^*\varphi = \pm i\varphi$ has no solutions $\varphi \neq 0$, then $T = T^*$.

Lecture 4 **Thm 55.1** (basic criterion for s.adjointness).

Let T be a sym. op. on $\mathcal{H}$. Then the followin 3 statements are $\equiv$!

(a) T is s.adj

(b) T is closed and $\ker(T^* \pm i) = \{0\}$

(c) $\operatorname{Ran}(T \pm i) = \mathcal{H}$

Proof: (a) ⇒ (b) has just been proved.

Suppose (b) holds: we will prove (c). Note first that Ran $(T-i)$ is dense. Otherwise $\exists$ non-zero $\psi \in \mathcal{H}$ s.t.

$((T-i)\varphi, \psi) = 0 \quad \forall \varphi \in D(T) = D(T-i)$. But

$= D(T^*+i)$

then $\psi \in D(T^*)$ and $T^*\psi = -i\psi$ contradicting

$\ker(T^*+i) = \{0\}$. But Ran $(T-i)$ is closed.

Indeed for $\varphi \in D(T)$

$$\begin{aligned}\|(T-i)\varphi\|^2 &= ((T-i)\varphi, (T-i)\varphi) \\ &= \|T\varphi\|^2 + \|\varphi\|^2 + i((\varphi, T\varphi) - (T\varphi, \varphi)) \\ &= \|T\varphi\|^2 + \|\varphi\|^2.\end{aligned}$$

Thus if $(T-i)\varphi_n \to g$, $\varphi_n \in D(T)$ then

φ_n is necessarily Cauchy, $\varphi_n \to \varphi$ for some φ.

Hence as T is closed $\varphi \in D(T)$ and $(T-i)\varphi = g$

Thus Ran$(T-i)$ is closed and hence Ran$(T-i) = \mathcal{H}$.

Similarly Ran $(T+i) = \mathcal{H}$.

Finally we show that (c) $\Rightarrow$ (a). Suppose $\psi \in D(T^*)$.

Then by (c) $\exists\ \phi \in D(T)$ st $(T+i)\phi = (T^*+i)\psi$ $\quad (*)$

But then $\forall\ \varphi \in D(T)$.

$$((T-i)\varphi, \psi) = (\varphi, (T^*+i)\psi) = (\varphi, (T+i)\phi)$$

$$= ((T-i)\varphi, \phi) \quad \text{(as } T \text{ is symm)}.$$

But then as $\mathrm{Ran}(T-i) = \mathcal{H}$, we must have

$\phi \equiv \psi$. Thus $\psi \in D(T)$ and $T\psi = T\phi = T^*\psi$ by $(*)$.

Thus $T^* \subset T$ and as $T \subset T^*$ by symmetry we

conclude that $T = T^*$

(Note: (c) + symmetry $\Rightarrow$ implies, in particular, that T is automatically closed.)

The above Th^m automatically implies the

following (exercise)

<u>Corollary (57.1)</u> Let T be a sym op. in $\mathcal{H}$. Then the following are $\equiv$:

(a) T is e.s.a

(b) $\ker(T^* \pm i) = \{0\}$

(c) $\mathrm{Ran}(T \pm i)$ are dense

A sym. op. T may have no. self adj. extensions, or it may have many s. adj. extensions. As we noted it has precisely one $\iff$ T is e.s.adj.

<u>Exple 58.1</u> (no. s. adj. extensions)

Let $T = i\frac{d}{dx}$ acting in $L^2(0,\infty)$

$$D(T) = \{ f \in L^2(0,\infty) : f \in AC,\ f' \in L^2(0,\infty),\ f(0)=0 \}$$

T is closed and symmetric. Indeed if

$$D(T) \ni f_n \to f.$$
$$i f_n' \to g.$$

Then

(58.1) $$i f_n(x) = i f_n(0) + i \int_0^x f_n'(t)\,dt$$

$$= i \int_0^x f_n'(t)\,dt$$

As $i f_n' \to g$ in $L^2(0,\infty)$, we certainly have $i\int_0^x f_n'(t) \to \int_0^x g(t)\,dt$ $\forall\ x \in (0,\infty)$. But then from (58.1) $f_n(x)$ converges for each x. As $f_n \to f$ in $L^2(0,\infty)$, for some subseq. $\{n_k\}$ we must have

$$f_{n_k}(x) \to f(x) \qquad a.e.\ x.$$

We conclude that $i f(x) = \int_0^x g(t)\,dt$

Thus $f \in AC$, $f' \in L^2$, $f(0) = 0$ and so $f \in D(T)$

and $Tf = g$.

If $f, h \in D(T)$ then for any x

$$\int_0^x \overline{if'}\,h = (\bar{f}h)(x) - (\bar{f}h)(0) + \int_0^x \bar{f}\,(ih')\,dx.$$

$$= \bar{f}(x)\,h(x) + \int_0^x \bar{f}\,ih'$$

i.e.

(59.1) $$\int_0^x \overline{Tf}\,h = \overline{f(x)}\,h(x) + \int_0^x \bar{f}\,Th$$

Letting $x \to \infty$ in (59.1) we conclude that

$$\lim_{x\to\infty} \overline{f(x)}\,h(x) = c \qquad \text{exists.}$$

But $$\left|\int_0^\infty \overline{f(x)}\,h(x)\right| \le \|f\|_{L^2}\,\|h\|_{L^2}.$$

and hence the only possible value for c is 0. We conclude that $(Tf, h) = (f, Th)$ and so T is symmetric.

Now (exercise)

$$D(T^*) = \{ f \in L^2(0,\infty) : f \in AC,\ f' \in L^2(0,\infty) \}$$

$$T^* f = i f', \qquad f \in D(T^*).$$

Thus if T has a s-adj. extension S, say, then

$$T \subset S = S^* \subset T^*.$$

∴ S is necessarily a restriction of $i\frac{d}{dx}$ to

$$D(S) \subset D(T^*), \quad Sf = if', \quad f \in D(S).$$

~~Now suppose $f \in D(S)$ and $(S-i)f = 0$~~

~~Then $i(f'-f) = 0 \Rightarrow f(x) = ce^{-x}$.~~

~~But $S = S^* \Leftrightarrow \ker(S-i) = \ker(S^*-i) = \{0\}$~~

Now let φ be any element of $D(S)$. Then $\varphi(x) - \varphi(0)e^{-x}$ clearly belongs to $D(T)$. But $D(T) \subset D(S)$ and so $\varphi(x) - \varphi(0)e^{-x} \in D(S)$. In particular $\varphi(0)e^{-x} \in D(S)$. Now $(S+i)e^{-x} = i\left(\frac{d}{dx} + 1\right)e^{-x}$

$= 0$. But $\ker(S+i) = \ker(S^*+i) = \{0\}$

as S is s.adj. Hence we must have $\varphi(0) = 0$. But

then $D(S) = D(T)$ and $S = T$. But $T^* \supsetneq T$

contradicting that $T = S$ is s.adj. Thus T cannot

have any s.adj. extensions

Remark: There is an intuitive explanation of the above example (see Reed-Simon II, Sect. X.1). If T had a s.adj. extension, S, say, then by the functional calculus for self-adjoint operators we could exponentiate S to obtain the unitary group $\{e^{-itS}, t \in \mathbb{R}\}$. For $\varphi_0 \in C_0^\infty(0,\infty) \subset D(T) \subset D(S)$, set

$$\varphi(t) = e^{-itS}\varphi_0.$$

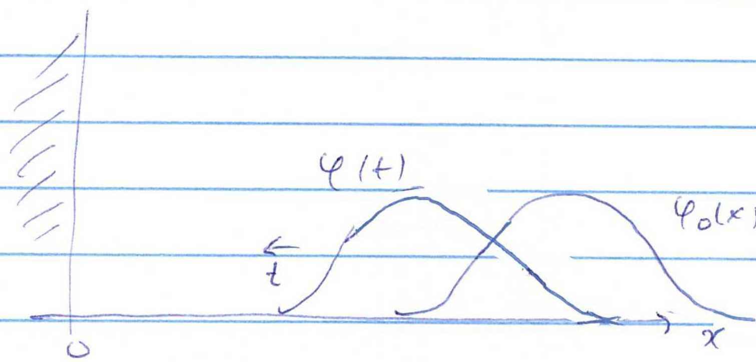
Then clearly

$$i\frac{d\varphi}{dt} = S\varphi(t) = i\frac{\partial\varphi}{\partial x}$$

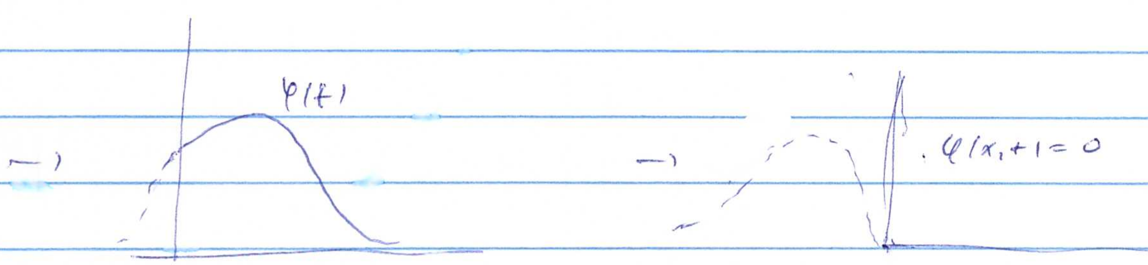
Thus

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial x} \quad \text{and so}$$

$$\varphi(x,t) = \varphi_0(x+t)$$



As time increases we see that the bump φ_0 moves to the left. Eventually for t large enough $\varphi(t)$ moves through the wall at $x=0$ and disappears



Thus $\|\varphi(t)\|_{L^2(0,\infty)} = 0$ for t suff. large. But this contradicts the unitarity of e^{-itS},

$$\|e^{-itS}\varphi_0\| = \|\varphi_0\| \neq 0 \quad \forall\, t \in \mathbb{R}\,!$$

On the other hand we have the following result.

Let

(63.0) $\left\{ \begin{array}{l} T = i\frac{d}{dx} \text{ with } D(T) = \{f \in L^2(0,1) : f \in AC[0,1], \\ \qquad f' \in L^2(0,1),\ f(0) = f(1) = 0\} \\ Tf = if' \text{ for } f \in D(T) \end{array} \right.$

as on p49 et seq. T is a closed, symmetric operator

and

$$D(T^*) = \{f \in L^2(0,1) : f \in AC[0,1],\ f' \in L^2(0,1)\}$$

$$T^*f = if' \text{ for } f \in D(T^*).$$

We now show that T has s.adj. extensions

and we compute all of them. If S is a s.adj

extension of T, then as before

$$T \subset S \subset T^*$$

so S is necessarily a restriction of T^*.

We show first that for some fixed $\alpha \in \mathbb{R}$

(63.1) $$D(S) \subset \{f \in L^2(0,1) : f \in AC[0,1],\ f' \in L^2,\ f(1) = e^{i\alpha} f(0)\}$$

Suppose that $f, g \in D(S)$. Then as S is certainly symmetric

$$(f, Sg) = (Sf, g)$$

As $S \subset T^*$, we must hat $Sg = ig'$, $Sf = if'$ and so

$$\int_0^1 \bar{f}\, ig' = \int_0^1 \overline{if'}\, g$$

But as $f, g \in AC[0,1]$, $\bar{f}g \in AC[0,1]$ and so we may integrate the LHS by parts to obtain

$$i\bar{f}g\Big|_0^1 + \int_0^1 \overline{if'}g = \int_0^1 \bar{f}\, ig'$$

and so

(64.1) $$\bar{f}(1)\, g(1) = \overline{f(0)}\, g(0) \qquad \forall f, g \in D(S)$$

Now $D(S)$ must contain a function $g(x)$ with either $g(0)$ or $g(1) \neq 0$: otherwise $D(S) = D(T)$

But then $S = T$ and so S is not s.adj, as $T^* \underset{\neq}{\supset} T$.

Choose $g \in D(S)$ st $|g(0)|^2 + |g(1)|^2 > 0$

and take $f = g$. From (64.1) we obtain

$$|g(1)|^2 = |g(0)|^2 .$$

and as $|g(1)|^2 + |g(0)|^2 > 0$ we must have

(65.1) $$g(1) = e^{i\alpha} g(0)$$

for some unique $e^{i\alpha}$, $\alpha \in \mathbb{R}$. But then for

any $f \in D(S)$ we have from (64.1)

$$\overline{f(1)}\, e^{i\alpha} g(0) = \overline{f(0)}\, g(0)$$

$$\Rightarrow \quad f(1)\, e^{-i\alpha} = f(0) \qquad (\text{as } g(0) \neq 0)$$

$$\Rightarrow \quad f(1) = e^{i\alpha} f(0)$$

This proves (63.1).

On the other hand, for any $\alpha \in \mathbb{R}$ let

(65.2) $$D(S_\alpha) = \{ f \in L^2(0,1) : f \in AC[0,1],\ f(1) = e^{i\alpha} f(0) \}$$
$$S_\alpha f = i f' , \quad f \in D(S_\alpha).$$

Then it is easy to verify that S_α is symmetric.

As $T \subset S_\alpha$, we must have $S_\alpha^* \subset T^*$

Let $f \in D(S_\alpha^*)$ so that

(66.1) $$(f, S_\alpha g) = (S_\alpha^* f, g) \quad \forall g \in D(S_\alpha).$$

Necessarily $f \in D(T^*) \subset AC[0,1]$ $S_\alpha^* f = if'$. Integrating by parts in (66.1) we have

$$(f, S_\alpha g) = \int \bar{f}\, i g' = \int \overline{if'}\, g + i\left(\bar{f}g(1) - \bar{f}g(0)\right)$$

$$= (S_\alpha^* f, g)$$

so necessarily

$$\bar{f}g(1) = \bar{f}g(0).$$

ie $$\bar{f}(1)e^{i\alpha} = \bar{f}(0)$$

ie $$f(1) = e^{i\alpha} f(0) \Rightarrow f \in D(S_\alpha)$$

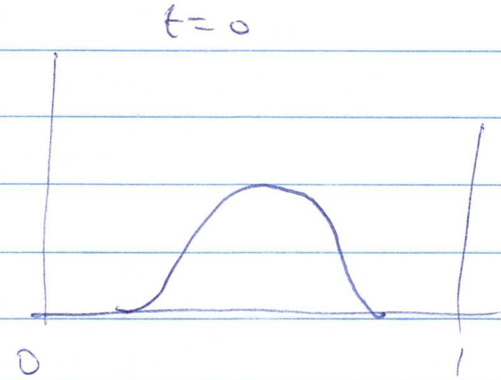
Thus $S_\alpha^* \subset S_\alpha$ and so S_α is s.adj.

Returning to (63.1), we see that $S \subset S_\alpha$

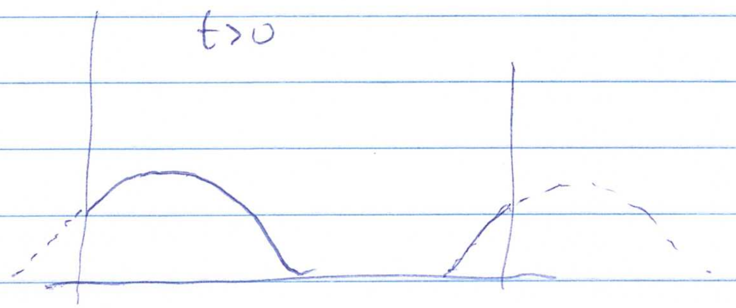
But s.adj. operators are maximally symmetric.
Indeed if $S=S^*$ and $S\subset M$ with M symmetric, then
$$S\subset M \Rightarrow M\subset M^*\subset S^* = S \quad \text{so } M=S.$$
In particular, as $S\subset S_\alpha$ we must have $S=S_\alpha$.
We conclude that every s.adj extension S of T is of the form S_α and conversely every S_α gives rise to a s.adj extension of T.

<u>Question</u> Why is it that $T = i\frac{d}{dx}$ acting on $L^2(0,\infty)$ with domain given on p 58 has no s.adj. extensions but $T = i\frac{d}{dx}$ on $L^2(0,1)$ with boundary condition $f(0)=f(1)=0$ as in (63.U) has s.adj. extensions?

It is simplest to consider the case $S_{\alpha=0}$ corresponding to the boundary condition $f(1)=f(0)$ i.e. periodic b.c's. For $\varphi_0(x)\in C_0^\infty(0,1)$ we can again consider $\varphi(t) = e^{-iS_{\alpha=0}t}\varphi_0 = \varphi_0(x+t)$



But now as t increases, what we lose through the left wall, we gain through the right wall by periodicity.



So again $\|\varphi(t)\|_{L^2(0,1)} = \|\varphi_0\|$!

The same is true for any S_α. But on $[0,\infty)$, there can be no compensating contribution

We recall the spectral Th^m for s.adj operators in multiplication operator form. We will not prove the Theorem in this course (see Reed-Simon Th^m

No! We will prove this Theorem in the next lecture! (2016, Spring)

§ VII.2 for bounded operators (and § VIII.3 for unbounded operators.) Our goal rather in this course (and also in the coming semester) to use this Theorem as a guide to understanding concretely the structure of particular s.adj. operator.

Th^{m} (sp.) (spec. Th^{m} - mult. oper. form).

Let A be a s.adj. oper. on a separ. Hilbert space $\mathcal{H}$ with domain $D(A)$. Then there exist measures $\{\mu_n\}_{n=1}^{N}$, $N = 1, 2, \ldots$, or ∞, on $\sigma(A)$ and a unitary operator

$$U : \mathcal{H} \to \bigoplus_{n=1}^{N} L^2(\mathbb{R}, d\mu_n)$$

$$\phi \mapsto \psi = U\phi = (\psi(\lambda,1), \psi(\lambda,2), \ldots)$$

$$\|\phi\|^2_{\mathcal{H}} = \sum_{n=1}^{N} \int |\psi(\lambda,n)|^2 \, d\mu_n(\lambda)$$

$\phi \in D(A) \iff \sum_{n=1}^{N} \int \lambda^2 |\psi(\lambda,n)|^2 \, d\mu_n(\lambda) < \infty$ where $\psi = U\phi$

and for $\phi = U^{-1}\psi \in D(A)$

$$(UAU^{-1}\psi)(\lambda, n) = \lambda\, \psi(\lambda, n) \quad . \quad \Box$$

The multiple measures $d\mu_1(\lambda)$, $d\mu_2(\lambda)$ reflect the multiplicity of the spectrum: if there is only 1 measure, we say that A is spectrally simple.

See Reed-Simon, § VII.6 for the commutative multiplicity thm which makes the notion of the multiplicity of a point $\lambda \in \sigma(A)$ precise.

Once we have the spec. Thm 69.1 we can define a functional calculus for A which associates to each bounded Borel function h on $\mathbb{R}$ an operator

(70.1) $$h(A) \equiv U^{-1} T_h U$$

$$(U h(A) U^{-1}\psi)(\lambda, n) = h(\lambda)\psi(\lambda, n)$$

The map $h \mapsto h(A)$ is a star-homomorphism i.e. $(\alpha h + \beta k)(A) = \alpha h(A) + k(A)$, $h(A) k(A) = hk(A)$, $h(A)^* =$

$\operatorname{Tr}(A)$, etc (see RSimon Thm VIII.5). Of particular interest is $e^{-itA} = h_t(A)$ where $h_t(\lambda) = e^{-it\lambda}$, $t \in \mathbb{R}$. Clearly $\{e^{-itA}\}$ is a strongly

continuous unitary group of commuting operators (see below)

As noted earlier in lecture 1, we may decompose each measure into a disjoint sum

$$d\mu_n = d\mu_{n,pp} + d\mu_{n,sing} + d\mu_{n,ac}$$

which gives rise to an orthogonal decomposition of $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{sing} \oplus \mathcal{H}_{ac}$$

such that each of the subspaces is invariant under A and $A \restriction \mathcal{H}_{pp}$ has spec. measures which are only pure point, $A \restriction \mathcal{H}_{sing}$ has spec. measures which are singular wrt Lebesgue measure, but have no points, and the spec.

measures for $A \restriction \mathcal{H}_{ac}$ are all purely abs. cont.
(see R.S. Chap VII)
To be more precise, if we consider $A_{ac} \equiv A \restriction \mathcal{H}_{ac}$,

for example, then A_{ac} is a s. adj. op. on the

Hilbert space $\mathcal{H}_{ac}$ and if we apply Theom 69.1 to

A_{ac} the measures $d\mu_1(A_{ac})$, $d\mu_2(A_{ac})$, ...

that we would obtain would all be abs. cont wrt.

Lebesgue measure, etc. To analyze an operator $A = A^*$ spectrally
means to determine the unitary U and the measures $d\mu_n$ & their decomp's into their pp, sing. & ac. parts
We now give some concrete examples of

the spectral theorem.

Exple 1

(72.1) Let

$$D(T) = \{ f \in L^2(\mathbb{R}) : f \in AC(\mathbb{R}) \}$$

$$Tf = i f'$$

Then our previous calculations — exercise! — show

that T is self-adjoint.

Let $\mathcal{F}f(s) = \lim \int e^{-isx} f(x) \frac{dx}{\sqrt{2\pi}} =$

$$\mathcal{F}^{-1}g(x) = \lim \int e^{ixs} g(s) \frac{ds}{\sqrt{2\pi}}$$

denote the Fourier - transform and inverse Fourier transform on $L^2(\mathbb{R})$. Both op.'s are unitary $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $\mathcal{F}^{-1} \circ \mathcal{F} = id$, $\mathcal{F} \circ \mathcal{F}^{-1} = id$.

Now for $f \in D(T)$

$$(\mathcal{F}^{-1} T f)(s) = \lim_{R \to \infty} \int_{-R}^{R} e^{+isx} i f'(x) \frac{dx}{\sqrt{2\pi}}.$$

A priori, the convergence as $R \to \infty$ is in $L^2(\mathbb{R})$. Now as e^{-isx} and $f(x)$ are $AC[-R, R]$ for any $R > 0$ we have

(74.0) $$\int_{-R}^{R} e^{+isx} i f'(x) \frac{dx}{\sqrt{2\pi}} = i \frac{e^{+isx} f(x)}{\sqrt{2\pi}} \Big|_{-R}^{R} - \int_{-R}^{R} is\, e^{+isx} i f(x) \frac{dx}{\sqrt{2\pi}}$$

Now observe that for any $x, x' \in \mathbb{R}$

(74.1) $$f^2(x) = f^2(x') + \int_{x'}^{x} 2 f' f \, dt$$

Now as $f, f' \in L^2$, $ff' \in L^1$ and so

$$2\int_{x'}^{x} f f'\, dt \quad \text{converges as} \quad x \to \infty .$$

But then from (74.1), $f^2(x)$ converges as $x \to \infty$, to c, say. But as $f \in L^2$, we must have $f^2(x) \to 0$ and hence $f(x) \to 0$ as $x \to \infty$ and similarly $-\infty$. Now we can choose a subsequence

$$R_n \to \infty \quad \text{st} \quad \int_{-R_n}^{R_n} e^{+isx}\, if' \frac{dx}{\sqrt{2\pi}} \to \mathcal{F}^{-1} if'(s)$$

$$\text{and} \quad +s\int_{-R_n}^{R_n} e^{+isx} f(x) \frac{dx}{\sqrt{2\pi}} \to +s\,\mathcal{F}^{-1} f(s) \quad \text{for a.e. } s \in \mathbb{R}.$$

We conclude that

$$\mathcal{F}^{-1} T f = s\, \mathcal{F}^{-1} f$$

Thus $U = \mathcal{F}^{-1}$, $d\mu(s) = ds$ and the spectral multiplicity is 1.

<u>Exercise</u> Simplify the above calculation by showing

first the $C_0^\infty(\mathbb{R})$ is a core for T ie

if $f \in D(T)$ then $\exists f_n \in C_0^\infty(\mathbb{R})$ st

$$f_n \to f \quad \text{an} \quad i f_n' \to Tf .$$

Exple 2

Let $\mathcal{H} = \ell^2(-\infty, \infty) = \{ a = \{a_n\}_{-\infty}^{\infty} : \sum_{-\infty}^{\infty} |a_n|^2 < \infty \}$

Let $L : \mathcal{H} \to \mathcal{H}$ st $(La)_n = a_{n+1}$

(why?)

ie L shifts to the left. $L^* = R$ with $(Ra)_n = a_{n-1}$

= shift to right. Let $A = R + L = L^* + L = A^*$.

which is a bded s. adj. operator

Now map $\mathcal{H}$ into $L^2[0,1]$ by

$$U : a_n \to \sum_{-\infty}^{\infty} a_n e^{2\pi i n x}$$

Then

$U L U^{-1}$ is mult. by $e^{-2\pi i x}$ in $L^2(0,1)$

and

$U R U^{-1}$ is mult. by $e^{2\pi i x}$ in $L^2(0,1)$