$$(UAx)(\lambda) = \left(U\left(\sum \lambda_i (u_i, x) u_i\right)\right)(\lambda)$$

$$= \lambda_i (u_i, x) \qquad \text{if } \lambda = \lambda_i$$
$$= 0 \qquad \text{otherwise}.$$

ie $\quad (UAU^{-1}f)(\lambda) = \lambda f(\lambda).$

<u>Exercise</u> Describe U when A has repeated eigenvalues. □

Instead of $\mu = \sum_{i=1}^{n} \delta(\lambda - \lambda_i)$ we could have chosen $\tilde{\mu} = \sum_{i=1}^{n} a_i \delta(\lambda - \lambda_i)$ for <u>any</u> $a_1, \ldots, a_n > 0$. This would have led to a unitary op. $\tilde{U} : \mathbb{C}^n \to L^2(\mathbb{R}, d\tilde{\mu})$ and again $(\tilde{U}A\tilde{U}^{-1}f)(\lambda) = \lambda f(\lambda)$

So we see that spectral reps of form (9.1)(9.2) are not unique: but we also see how they may differ!

<u>Lecture 2</u>

In infinite dimensions the situation is complicated by the following 2 observations:

(19.1) • we would like to consider symmetric operators. Such operators would (should?) have a spectral Th^m and give rise to unitary dynamics e^{-itA}, for example

• on the other hand the natural operators are unbounded.

eg consider the multiplication operator X

on $L^2(\mathbb{R}, dx)$, $Xf(x) = x f(x)$. Clearly X is symmetric

Let $\chi_n(x) = \frac{1}{\sqrt{n}} \chi_{[0,n]}(x)$

where $\chi_{[0,n]}$ is the characteristic function of $[0,n]$

Then clearly $\|\chi_n\|_{L^2(\mathbb{R})} = 1$ but

$$\|X\chi_n\|^2 = \frac{1}{n}\int_0^n x^2 = \frac{n^2}{n \times 3} \to \infty \text{ as } n \to \infty$$

ie we cannot have $\|X\chi_n\|_{L^2} \leq M\|\chi_n\|_{L^2}$

for some $M < \infty$ ie X is not bounded.

Now there is a fundamental Theorem of

Hellinger and Toeplitz that says the following:

An everywhere defined symmetric operator is automatically bounded.

To prove this result, let A be symmetric and everywhere defined, i.e. $D(A) = \mathcal{H}$, and consider the graph of A

$$\Gamma(A) = \{\langle x, Ax\rangle : x \in \mathcal{H}\} \subset \mathcal{H} \times \mathcal{H}$$

$\mathcal{H} \times \mathcal{H}$ has topology $\|\langle x, y\rangle\| = \|x\|_{\mathcal{H}} \oplus \|y\|_{\mathcal{H}}$.

The closed graph Th^m says that if $\Gamma(A)$ is closed in $\mathcal{H} \times \mathcal{H}$, then A is bounded. So suppose $\langle x_n, Ax_n\rangle \to \langle x, y\rangle \in \mathcal{H} \times \mathcal{H}$. Then from symmetry, for any $\varphi \in \mathcal{H}$

$$(x_n, A\varphi) = (Ax_n, \varphi)$$

$$\Rightarrow \quad (x, A\varphi) = (y, \varphi)$$

ie $\quad (y, \varphi) = (Ax, \varphi) \qquad \forall \varphi$

$$\Rightarrow \quad y = Ax$$

Thus $\langle x, y\rangle = \langle x, Ax\rangle \in \Gamma(A)$ and so A is bounded.

We conclude from these considerations that our natural operators cannot be defined on all of $\mathcal{H}$. We will always need to find appropriate domains $D(A)$ for the operators, $D(A) \subsetneq \mathcal{H}$.

(21.1) Definition An operator $A : \mathcal{H} \to \mathcal{H}$ is a linear map from its domain $D(A)$, a linear subspace of $\mathcal{H}$, into $\mathcal{H}$. In addition, we always assume $D(A)$ is dense in $\mathcal{H}$.

Example X as above with $D_0(X) = C_0^\infty(\mathbb{R})$ is a densely defined symmetric operator in $\mathcal{H} = L^2(\mathbb{R})$.

This is also true if we take the larger domain $D_S(X) = \mathcal{S}(\mathbb{R})$. Moreover, this is true if we take

(21.2) $$D_{max}(X) = \{ f \in L^2(\mathbb{R}) : x f(x) \in L^2(\mathbb{R}) \}$$

$D_{max}(X)$ is clearly the maximal domain for X

to be a well-defined symmetric operator. As we will see later, X is self-adjoint on $D_{max}(X)$.

A bounded operator A on $\mathcal{H}$ has the property that

(22.1) if $x_n \to x$, then $\{Ax_n\}$ is Cauchy, and it converges to Ax

It turns out that there is a property of operators that is weaker than boundedness, but still very viable! ~~fortunately~~ our natural operators have this property, once their domains are chosen appropriately. The property is closedness.

Definition 22.1 We say that an operator $T : D(T) \to \mathcal{H}$ is closable if for $\varphi_n \in D(T)$

$$\varphi_n \to 0, \quad T\varphi_n \to y$$

$$\Rightarrow \quad y = 0$$

<u>Definition 23.1</u> We say that an operator T_1 is an <u>extension</u> of T, written $T_1 \supset T$ if

$$D(T_1) \supset D(T) \quad \text{and} \quad T_1\varphi = T\varphi \quad \forall\, \varphi \in D(T) \subset D(T_1)$$

<u>Definition 23.2</u>

We say that an operator T is <u>closed</u> if for

$\varphi_n \in D(T)$

$$\varphi_n \to f \quad , \quad T\varphi_n \to g$$

$$\Rightarrow \quad f \in D(T) \quad \text{and} \quad Tf = g$$

Note that if T is closed, then it is closable for if

$$D(T) \ni \varphi_n \to 0 = f$$
$$T\varphi_n \to g$$

then as 0 always lies in $D(T)$, we have $f = 0 \in D(T)$, and $g = Tf = T0$. But $T0 = 0$ (why?) and so $g = 0$. Thus T is closable.

Note also that every bounded operator is always closed and hence also closable. This is because a bounded, densely defined operator, can always be extended to a bdd operator on all of $\mathcal{H}$ and if $x_n \to x$, $Tx_n \to y$, then by continuity $Tx = y$.

Note finally that T is a closed operator if and only if the graph of T, $\Gamma(T) = \{\langle x, Tx \rangle : x \in D(T)\}$ is closed in $\mathcal{H} \times \mathcal{H}$. Thus, by the closed graph theorem, if T is closed and $D(T) = \mathcal{H}$, then T is bounded.

If an operator T is closable, then we can define its <u>closure</u>, $\bar{T}$, as follows:

$D(\bar{T}) = \{ f : \exists\ D(T) \ni \varphi_n \to f,\ T\varphi_n \text{ is Cauchy} \}$

Then for $f \in D(\bar{T})$, $\bar{T} f \equiv g$, where $g = \lim_{n\to\infty} T\varphi_n$.

To verify that $\bar{T}$ is well-defined, suppose that for some other sequence

$$D(T) \ni \psi_n \to f, \quad T\psi_n \to G$$

We must show $g = G$. Now $D(T) \ni \varphi_n - \psi_n \to 0$ and $T(\varphi_n - \psi_n) = T(\varphi_n) - T(\psi_n) \to g - G$. Hence as T is closable $g - G = 0$ ie $g = G$.

Clearly $\bar{T}$ is linear and $\bar{T} \supset T$ (why?). Also we have the following result

Proposition 25.1 Let T be a closable operator in $\mathcal{H}$. Then

(a) $\bar{T}$ is a closed operator

(b) $\bar{T}$ is the smallest closed extension of T i.e. if $\tilde{T} \supset T$ and $\tilde{T}$ is closed, then $\tilde{T} \supset \bar{T}$.

Proof: (a) Suppose $\varphi_n \in D(\bar{T})$, $\varphi_n \to f$, $\bar{T}\varphi_n \to g$. Must show $f \in D(\bar{T})$ and $\bar{T}f = g$.

Now for each n, $\exists$ $\psi_n \in D(T)$ st

$$\|\varphi_n - \psi_n\| < \frac{1}{n}, \quad \|\bar{T}\varphi_n - T\psi_n\| < \frac{1}{n}$$

Thus

$$\|\psi_n - f\| \le \|\psi_n - \varphi_n\| + \|\varphi_n - f\| \to 0$$

$$\|T\psi_n - g\| \le \|T\psi_n - \bar{T}\varphi_n\| + \|\bar{T}\varphi_n - g\| \to 0$$

$\therefore$ $f \in D(\bar{T})$ and $\bar{T}f = g$, as desired

(b) Suppose $f \in D(\bar{T})$ and $\bar{T}f = g$. Then $\exists$ $\varphi_n \in D(T)$

st $D(T) \ni \varphi_n \to f$ and $T\varphi_n \to g$. But $T \subset \tilde{T}$

$\therefore$ $D(\tilde{T}) \ni \varphi_n \to f$, $\tilde{T}\varphi_n = T\varphi_n \to g$ and so

as $\tilde{T}$ is closed, $f \in D(\tilde{T})$ and $\tilde{T}f = g = \bar{T}f$

Hence $\tilde{T} \supset \bar{T}$.

Exple 26.1

Let X denote mult. by x in $L^2(\mathbb{R})$ with

domain $D_{max}(X) = \{f \in L^2(\mathbb{R}) : x f(x) \in L^2(\mathbb{R})\}$

as in (21.2). Then X is closed.

Indeed, suppose $D_{max} \ni \varphi_n \to f$ and $X\varphi_n \to g$

Now for any $R>0$

$$\int_{-R}^{R} x^2 |f(x)|^2 dx = \lim_{n\to\infty} \int_{-R}^{R} x^2 |\varphi_n(x)|^2 \quad \text{(why)}$$

$$= \int_{-R}^{R} |g(x)|^2 dx \leq \|g\|^2$$

Letting $R\to\infty$, we see that $\int x^2 |f(x)|^2 dx < \infty$

and so $f \in D_{max}(X)$. By defn., we then have

$X f(x) = x f(x)$.

Again for any $R>0$, $\int_{-R}^{R} |Xf - g|^2 = \int_{-R}^{R} |x f(x) - g|^2$

$$= \lim_{n\to\infty} \int_{-R}^{R} |x \varphi_n(x) - g| \leq \lim_{n\to\infty} \int_{-\infty}^{\infty} |x\varphi_n(x) - g(x)|^2 dx \to 0$$

Hence, as $R>0$ is arbitrary, we have $Xf = g$. Thus

X is closed on $D_{max}(X)$.

<u>Exple 27.1</u>

Let $\mathcal{H} = L^2(\mathbb{R})$, $D(T) = C_0^\infty(\mathbb{R})$ and

$$Tf(x) = f'(x), \quad f \in D(T).$$

Clearly T is unbounded (why?)

We show

(28.1) $\quad T$ is closable

(28.2) $\quad D(\bar{T}) = \{ f \in L^2 : Df(x) \in L^2 \}$

$$\bar{T}f = f' \quad \text{for } f \in D(\bar{T})$$

Here $Df(x)$ ~~refers~~ to the weak derivative of f i.e. $Df(\varphi)$ $= -\int f \varphi' \;\; \forall \varphi \in C_0^\infty(\mathbb{R})$, (which is defined as for all distributions). Thus the meaning of "$Df(x) \in L^2$" is that $Df(\varphi)$ is given by the L^2 function $Df(x)$ i.e.

$$Df(\varphi) = \int Df(x)\,\varphi \quad \text{for some } Df(x) \in L^2$$

Proof of (28.1): ~~Note that if $f \in C_0^\infty(\mathbb{R})$ and φ is any element in $C_0^\infty(\mathbb{R})$, $Df(\varphi) = -\int f\varphi' = \int f'\varphi$, by integ. by parts. Hence $f \in D(\bar{T})$ and $\bar{T}f = f'$~~

Suppose $C_0^\infty(\mathbb{R}) \ni \varphi_n \to 0$, $T\varphi_n \to g \in L^2$.
Must show $g = 0$. For any $\varphi \in C_0^\infty(\mathbb{R})$, integration by parts $\Rightarrow$
$\int \varphi (T\varphi_n) = \int \varphi \varphi_n' = -\int \varphi' \varphi_n$.

Letting $n \to \infty$, get $\int \varphi g = -\int \varphi' 0 = 0$. Hence $g = 0$

Thus T is closable.

Proof of (28.2)

Let $D(\tilde{T}) = \{ f \in L^2 : Df(x) \in L^2 \}$

$\tilde{T} f = Df(x)$ for $f \in D(\tilde{T})$

We want to show $\tilde{T} = \bar{T}$.

First we show that $\bar{T} \subset \tilde{T}$. Let $f \in D(\bar{T})$

Then $\exists\ \varphi_n \in C_0^\infty(\mathbb{R})$, $\varphi_n \to f$, $T\varphi_n = \varphi_n' \to \bar{T} f$

Now for any $\varphi \in C_0^\infty$, $\int \varphi \varphi_n' = -\int \varphi' \varphi_n$. Letting

$n \to \infty$, we obtain $\int \varphi \bar{T} f = -\int \varphi' f = Df(\varphi)$

ie the weak deriv. Df of f is given by an L^2

function $\bar{T} f$. By defn then $f \in D(\tilde{T})$ and $\tilde{T} f = Df(x)$

$= \bar{T} f$ $\therefore$ $\bar{T} \subset \tilde{T}$. Conversely, suppose that f

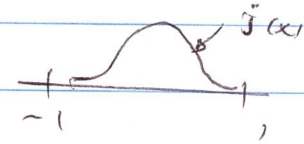
$\in D(\tilde{T})$. Then $\exists\ Df(x) \in L^2$ st

(29.1) $$\int f \varphi' = -\int Df(x)\, \varphi \quad \forall\ \varphi \in C_0^\infty$$

Here $Df(x) = \tilde{T} f$.

Let $j(x)$ be any pos. func. in C_0^∞ with support in

$(-1,1)$ and $\int_{\mathbb{R}} j(x)\,dx = 1$



Define $j_\varepsilon(x) \equiv \varepsilon^{-1} j(x/\varepsilon)$, $\varepsilon > 0$.

Let $h(x)$ be a pos. function in C_0^∞ with $h(0) = 1$

and set

$$f_\varepsilon(x) = \int j_\varepsilon(x-t)\, h(\varepsilon t)\, f(t)\, dt$$

A standard calculation (exercise) shows that $f_\varepsilon \in C_0^\infty$

and $f_\varepsilon \to f$ in $L^2(\mathbb{R})$. Also

$$f_\varepsilon'(x) = \int j_\varepsilon'(x-t)\, h(\varepsilon t)\, f(t) = -\int \frac{\partial}{\partial t} j_\varepsilon(x-t)\, h(\varepsilon t)\, f(t)$$

$$= -\int \frac{\partial}{\partial t}\left(j_\varepsilon(x-t)\, h(\varepsilon t)\right) f(t)\, dt$$

$$+ \varepsilon \int j_\varepsilon(x-t)\, h'(\varepsilon t)\, f(t)\, dt$$

$$= \int j_\varepsilon(x-t)\, h(\varepsilon t)\, Df(t) + \varepsilon \int j_\varepsilon(x-t)\, h'(\varepsilon t)\, f(t)$$

(by (29.1) as $j_\varepsilon(x-t)\, h(\varepsilon t) \in C_0^\infty$)

$$= \mathrm{I} + \mathrm{II}$$

By the above exercise, $\mathrm{I} \to Df(x)$ in L^2. As

$$\|\mathrm{II}\|_{L^2}^2 \le \varepsilon^2 \int \left|\int j_\varepsilon(x-t)\, h'(\varepsilon t)\, f(t)\, dt\right|^2 dx .$$

$$\leq \varepsilon^2 \int \Big| \underbrace{\int j_\varepsilon(x-t)\,dt}_{1} \Big| \Big| \int j_\varepsilon(x-t)\, |h'(\varepsilon t)|^2 |f(t)|^2\, dt \Big|\, dx$$

$$= \varepsilon^2 \int |h'(\varepsilon t)|^2 |f(t)|^2\, dt$$

$$\leq \|h'\|_{L^\infty}^2\, \varepsilon^2\, \|f\|^2 \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

$\therefore\ f_\varepsilon' \to Df(x)$ in L^2 as $\varepsilon \downarrow 0$. Hence $f \in D(\bar{T})$ and $\bar{T}f = Df(x)$. So $\tilde{T} \subset \bar{T}$ and $\tilde{T} = \bar{T}$

Important remark:

Recall that a function $f : [a,b] \to \mathbb{R}$, $a<b$, is absolutely continuous (AC) if for every $\varepsilon > 0$ $\exists\ \delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

for every finite collection of non-overlapping intervals $\{(a_i, b_i)\}$ in $[a,b]$ with

$$\sum_{i=1}^{n} |b_i - a_i| < \delta.$$

Clearly f AC $\Rightarrow$ f is continuous.

The Fundamental Theorem of Calculus says the following: If f is AC, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\exists$ for a.e. x, $f' \in L^1([a,b])$ and for $a \le x \le b$

$$f(x) = \int_a^x f'(t)\,dt + f(a)$$

Conversely if for $a \le x \le b$

$$f(x) = \int_a^x g(t)\,dt + f(a)$$

for some $g \in L^1([a,b])$, then f is AC and $f'(x) = g(x)$ a.e.

We say $f \in$ AC on $\mathbb{R}$ if f is AC for every interval $[a,b]$, $-\infty < a < b < \infty$. In 1-dim, we have the following result.

Claim 32.1

$$\{f \in L^2(\mathbb{R}) : \underbrace{D f(x)}_{\text{distrib. deriv}} \in L^2(\mathbb{R})\}$$

$$= \{f \in L^2(\mathbb{R}) : f \text{ AC}, \; f' \in L^2(\mathbb{R})\}$$

Here f' is the a.e. limit $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Also $f'(x) = Df(x)$

Suppose $f \in \{f \in L^2 : Df \in L^2\}$ Then

$$\int \varphi' f = -\int \varphi Df \qquad \forall \varphi \in C_0^\infty$$

Integrating RHS by parts we obtain

$$\int \varphi' f = \int \varphi'(x) \int_0^x Df(t)\, dt$$

i.e.

(33.1) $\int \varphi'(x) g(x)\, dx = 0 \qquad \forall \varphi \in C_0^\infty$ where

$g(x) = f(x) - \int_0^x Df(t)\, dt \in L^2_{loc}(\mathbb{R})$. Now for

any $\psi(x) \in C_0^\infty$, let

$$\varphi(x) \equiv \int_{-\infty}^x \psi(t)\, dt - \chi(x) \int_{-\infty}^\infty \psi(t)\, dt$$

where χ is any C^∞ function such that

$$\chi(x) = 0 \quad \text{for } x < 0$$
$$\chi(x) = 1 \quad \text{for } x > 1.$$

Clearly $\varphi \in C_0^\infty(\mathbb{R})$ and we may insert it

into (33.1) we obtain

$$\int \left(\psi(x) - \chi'(x) \int_{-\infty}^\infty \psi(t)\, dt\right) g(x) = 0$$

ie $0 = \int \psi(x) g(x) - \left(\int \psi(t)\, dt\right) \int \chi'(x) g(x)\, dx$

$$= \int \varphi(x) \left(g(x) - \int_{-\infty}^{\infty} \chi'(t) g(t)\, dt \right) dx$$

As $\varphi \in C_0^\infty$ is arbitrary, we see that

$$g(x) = \text{const} = \int \chi' g(t)\, dt .$$

$$\therefore \quad f(x) = \int_0^x Df(t)\, dt + \text{const}$$

and so f is AC and $f'(x) = Df(x)$ a.e.

This shows that

(34.1) $$\{f \in L^2 : Df \in L^2\} \subset \{f \in L^2 : f \text{ AC}, f' \in L^2\}$$

and $Df(x) = f'(x)$

Conversely suppose $f \in L^2$, f AC with $f' \in L^2$. Then for any $\varphi \in C_0^\infty$, $\varphi f \in$ AC and so for any $a < b$ and $\operatorname{supp} \varphi \subset [a, b]$,

$$0 = \varphi(b) f(b) - \varphi(a) f(a) = \int_a^b (\varphi f)'\, dx = \int_a^b \varphi' f + \varphi f'$$

Thus,

$$\int_a^b \varphi' f = -\int_a^b \varphi f'$$

Thus $Df \in L^2$ and $Df = f'$. This establishes the reverse equality in (34.1) and hence the Claim 32.1.

We conclude from the above that if

$$Tf = f', \qquad D(T) = C_0^\infty$$

then $D(\bar{T}) = \{ f \in L^2 : f \text{ ac}, f' \in L^2 \}$

$$\bar{T} f = f'$$

Lecture 3

Definition 35.1 : adjoint operator T^*

Let T be a densely defined operator in $\mathcal{H}$. Let $D(T^*)$ be the set of $\varphi \in \mathcal{H}$ for which $\exists\, \eta \in \mathcal{H}$ st

(35.2) $$(T\psi, \varphi) = (\psi, \eta) \qquad \forall\, \psi \in D(T)$$

Then for $\varphi \in D(T^*)$,

(35.3) $$T^* \varphi = \eta$$

Note that as T is densely defined, η is unique in (35.2) and hence T^* is well-defined.

~~Note: $\varphi \in D(T^*)$ if and only if $(T\psi, \varphi)$ is a bdd anti-lin. funct. wrt ψ.~~