closed operator, q is a closed form.

Hence $\exists$ a s.adj. operator T with

$D(T) \subset Q(q) = D(A)$ such that $\psi \in D(T) \iff$

$q(\varphi, \psi) = (\varphi, h)$, $\forall \varphi \in D(A)$ for some

$h \in \mathcal{H}$. In this case, $h = T\psi$.

But if $q(\varphi, \psi) = (A\varphi, A\psi) = (\varphi, h)$. $\forall \varphi \in D(A)$

then $A\psi \in D(A^*)$ and $A^*A\psi = h = T\psi$.

Hence $D(T) \subset D(A^*A)$ and $T\psi = A^*(A\psi)$
if $\psi \in D(T)$.

Conversely if $\psi \in D(A^*A)$ then $A\psi \in D(A^*)$

and $q(\varphi, \psi) = (A\varphi, A\psi) = (\varphi, A^*A\psi)$ $\forall \varphi \in D(A)$

Thus $\psi \in \operatorname{Dom} T$ and $T\psi = A^*A\psi$ Thus $D(A^*A) \subseteq$

$D(T) \Rightarrow \operatorname{Dom} T = D(A^*A)$ and $T\psi = A^*A\psi$ on $D(T)$.

Lecture 12

Exple 3

Let $\mathcal{H} = L^2(0,1)$ and let $V(x) \in L^1(dx)$, V real valued.

Set

$$q(f,g) = \int_0^1 \bar{f}' g' + \int_0^1 V(x) \bar{f}(x) g(x).$$

with $Q(q) = \{ f \in AC[0,1] : f' \in L^2(0,1) \}$

From (137.1) we see that for any $\varepsilon > 0$

(151.1) $$\left| \int_0^1 V(x) |f(x)|^2 \right| \leq \left(\int_0^1 |V(x)| dx \right) \left((1+\varepsilon^{-1}) \int_0^1 |f|^2 + \varepsilon \int_0^1 |f'|^2 \right.$$

It follows from (151.1) (exercise) q is closed on $Q(q)$. Let T be the s.adj operator associated with q. Then (exercise).

(151.2) $$D(T) = \{ f \in AC^2[0,1] : -f'' + Vf \in L^2 \\ f'(0) = f'(1) = 0 \}$$

$$Tf = -f'' + Vf$$

Notice that as $V \in L^1(0,1)$ we do not know a priori that T is even defined for vectors f other than zero. Nevertheless $D(T)$ is dense in $L^2(0,1)$ and T is s.adj. on $D(T)$.

We now describe a very useful idea in spectral theory — the <u>min-max</u> principle.

The idea is the following. Suppose A & B a 2 3×3 s.adj. matrices with $A \le B$ in the sense that $(\varphi, A\varphi) \le (\varphi, B\varphi)$ $\forall \varphi \in \mathbb{C}^3$. Let $\lambda_1(A) \le \lambda_2(A) \le \lambda_3(A)$ be the eigenvalues of A listed in increasing order. Similarly $\lambda_1(B) \le \lambda_2(B) \le \lambda_3(B)$ for the eig's of B. We expect $\lambda_i(A) \le \lambda_i(B)$ for $i = 1,2,3$. How to prove it?

Let ψ_1, ψ_2, ψ_3 be the 3 eigenvectors of A chosen to be orthonormal. Writing $\psi = \alpha_1\psi_1 + \alpha_2\psi_2 + \alpha_3\psi_3$ we see that

$$\frac{(\psi, A\psi)}{(\psi,\psi)} = \frac{\lambda_1(A)|\alpha_1|^2 + \lambda_2(A)|\alpha_2|^2 + \lambda_3(A)|\alpha_3|^2}{|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2}$$

so $\displaystyle\min_{\psi\neq 0} \frac{(\psi, A\psi)}{(\psi,\psi)} = \lambda_1(A)$ and $\displaystyle\max_{\psi\neq 0} \frac{(\psi, A\psi)}{(\psi,\psi)} = \lambda_3(A)$, with similar formulae for $\lambda_1(B)$ and $\lambda_3(B)$. As $(\psi, A\psi) \le (\psi, B\psi)$ we see immediately that $\lambda_1(A) \le \lambda_1(B)$ and $\lambda_3(A) \le \lambda_3(B)$. But what about $\lambda_2(A) \le \lambda_2(B)$?

First note that

$$\lambda_2 = \min_{\substack{(\alpha_1,\alpha_2,\alpha_3) \neq 0 \\ \alpha_1 = 0}} \frac{|\alpha_1|^2\lambda_1 + |\alpha_2|^2\lambda_2 + |\alpha_3|^2\lambda_3}{|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2}$$

ie

$$\lambda_2(A) = U_A(\psi_1)$$

where

$$U_A(\psi) = \min_{\{\varphi : (\psi,\varphi) = 0,\ \varphi \neq 0\}} \frac{(\varphi, A\varphi)}{(\varphi,\varphi)}.$$

On the other hand, for any ψ, $\exists$ $\varphi = \alpha_1\psi_1 + \alpha_2\psi_2$, $(\alpha_1, \alpha_2) \neq 0$ st $(\varphi, \psi) = 0$. For such φ

$$\frac{(\varphi, A\varphi)}{(\varphi,\varphi)} = \frac{|\alpha_1|^2\lambda_1 + |\alpha_2|^2\lambda_2}{|\alpha_1|^2 + |\alpha_2|^2} \le \lambda_2$$

and so

$$U_A(\psi) \le \lambda_2$$

As

$$U_A(\psi_1) = \lambda_2$$

Thus

(154.1) $$\lambda_2(A) = \sup_{\psi} \min_{\substack{\varphi \perp \psi \\ \varphi \neq 0}} \frac{(\varphi, A\varphi)}{(\varphi, \varphi)}$$

It follows immediately from (154.1) and its B-analog, that $\lambda_2(A) \le \lambda_2(B)$.

The general result is as follows.

<u>Th^m (154.2)</u> (min-max principle, operator form)

Let H be a s.adj. oper. that is bdd below, i.e. $H \geq c$ for some c. Define

(154.3) $$\mu_n = \sup_{\varphi_1, \dots, \varphi_{n-1}} U_H(\varphi_1, \dots, \varphi_n)$$

where

(154.4) $$U_H(\varphi_1, \dots \varphi_m) = \inf_{\substack{\psi \in D(H),\, \|\psi\|=1 \\ \psi \perp \langle \varphi_1, \dots, \varphi_m \rangle}} (\psi, H\psi)$$

where $\langle \varphi_1, \dots, \varphi_m \rangle \equiv \operatorname{span}\{\varphi_i\}_{i=1}^m$.

Then for each fixed n,

(154.5) $$\mu_n(H) \le \inf\{\lambda : \lambda \in \sigma_{ess}(H)\}$$

and <u>if</u>

(a) $\mu_n(H)$ is below the essentual spectrum of H

i.e. $\mu_n(H) < \inf\{\lambda : \lambda \in \sigma_{ess}(H)\}$,

then $\mu_n(H)$ is the n^{th} eigenvalue of H, counting multiplicity

and if

(b) μ_n is the bottom of the essential spectrum i.e.

$\mu_n = \inf\{\lambda : \lambda \in \sigma_{ess}(H)\}$ then

$\mu_n = \mu_{n+1} = \mu_{n+2} = \cdots$ and there are at most

$n-1$ eigenvalues (counting multiplicity) below μ_n.

Recall:
- $\lambda \in \sigma(A) \iff P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) \neq 0 \quad \forall \varepsilon > 0$
- $\lambda \in \sigma_{ess}(A) \iff \dim P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) = \infty \quad \forall \varepsilon > 0$
- $\lambda \in \sigma_{disc}(A) \iff$ $\lambda \in \sigma(A)$ and $\dim P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A) < \infty$ for some $\varepsilon > 0$.

Proof: Note first that $\mu_1, \mu_2, \ldots$ is a non-decreasing sequence. Indeed for any $\psi_1, \ldots, \psi_m$, we clearly have

$$U_H(\varphi_1, \dots, \varphi_{m-1}) \le U_H(\varphi_1, \dots, \varphi_{m-1}, \varphi_m)$$

and so

$$\mu_m = \sup_{\varphi_1, \dots, \varphi_{m-1}} U_H(\varphi_1, \dots, \varphi_{m-1})$$

$$\le \sup_{\varphi_1, \dots, \varphi_{m-1}} U_H(\varphi_1, \dots, \varphi_{m-1}, \varphi_m)$$

$$\le \sup_{\varphi_1, \dots, \varphi_{m-1}, \varphi_m} U_H(\varphi_1, \dots, \varphi_m) = \mu_{m+1}.$$

Now let P_Ω be the projection-valued measure for H.

We first prove that

(156.1) $$\dim\left[\operatorname{Ran} P_{(-\infty, a)}\right] < n \qquad \text{if } a < \mu_n.$$

(156.2) $$\dim\left[\operatorname{Ran} P_{(-\infty, a)}\right] \ge n \qquad \text{if } a > \mu_n.$$

Suppose (156.1) is false. Then we can find an n-dim space $V \subset D(H)$ so that for any $\varphi \in V$, $(\varphi, H\varphi) < a\|\varphi\|^2$. That $V \subset D(H)$ is a consequence of the fact that H is bded below which implies $\operatorname{Ran} P_{(-\infty, a)} \subset D(H)$ if $a < \infty$. But then given any $\varphi_1, \dots, \varphi_{n-1}$ we can find $\varphi \in V \cap \langle \varphi_1, \dots, \varphi_{n-1} \rangle^\perp$. Thus $U_H(\varphi_1, \dots, \varphi_{n-1}) \leq a$ for any $\varphi_1, \dots, \varphi_{n-1}$ so $\mu_n(H) \leq a$, which is a contradiction. This proves (156.1).

Now suppose (156.2) is false. Then $\dim(P_{(-\infty, a)}) \leq n-1$ so we can find $\varphi_1^{(0)}, \dots, \varphi_{n-1}^{(0)}$ with $\langle \varphi_1^{(0)}, \dots, \varphi_{n-1}^{(0)} \rangle = \operatorname{Ran}(P_{(-\infty, a)})$. Then any $\psi \in \langle \varphi_1^{(0)}, \dots, \varphi_{n-1}^{(0)} \rangle^\perp \cap D(H)$ is in $\operatorname{Ran} P_{[a, \infty)}$, so $(\varphi, H\varphi) \geq a \|\varphi\|^2$. Therefore $U_H(\varphi_1^{(0)}, \dots, \varphi_{n-1}^{(0)}) \geq a$ and $\mu_n \geq a$, which is a contradiction. This proves (156.2).

Note that if $\mu_n = \infty$ for some n, then by (156.1), $\dim \operatorname{Ran}(P_{(-\infty, a)}) < n$ $\forall a$ which is impossible as $P_{(-\infty, a)} \to$ identity as $a \to +\infty$. Thus $\mu_n < \infty$ $\forall n$.

Case I $\dim(P_{(-\infty,\mu_n+\varepsilon)}) = \infty$ $\forall\, \varepsilon > 0$. We claim that we are in the situation (b) in the Theorem. For by (156.1), $\dim(P_{-\infty,\mu_n-\varepsilon}) \le n-1$ and therefore $\dim(P_{\mu_n-\varepsilon,\mu_n+\varepsilon}) = \infty$ $\forall\, \varepsilon > 0$. Thus $\mu_n \in \sigma_{ess}(H)$. On the other hand, again by (156.1), if $a < \mu_n$ and $0 < \varepsilon < \mu_n - a$, then $\dim P_{(a-\varepsilon,a+\varepsilon)} < n < \infty$, so $a \notin \sigma_{ess}(H)$. Thus $\mu_n = \inf\{\lambda : \lambda \in \sigma_{ess}(H)\}$. Now $\mu_{n+1} \ge \mu_n$. If $\mu_{n+1} > \mu_n$, then $\dim P_{(-\infty,\frac{1}{2}(\mu_n+\mu_{n+1}))} \le n$, contradicting $\dim(P_{-\infty,\mu_n+\varepsilon}) = \infty$ $\forall\, \varepsilon > 0$. Thus $\mu_{n+1} = \mu_n$, etc. Finally, we note that if there were n eigenvalues strictly below μ_n, and a was the n^{th} eigenvalue, then $\dim P_{(-\infty,\frac{1}{2}(\mu_n+a))} \ge n$, contradicting (156.1). Thus situation (b) holds.

Case II $\dim P_{(-\infty,\mu_n+\varepsilon_0)} < \infty$ for some $\varepsilon_0 > 0$. We claim that we are then in situation (a) in the Theorem.

For, by (156.1) (156.2), for any $\varepsilon > 0$,

$$\dim\left(P_{(\mu_n-\varepsilon,\, \mu_n+\varepsilon)}\right) = \dim P_{(-\infty,\, \mu_n+\varepsilon)} - \dim P_{(-\infty,\, \mu_n-\varepsilon]}$$

$$\geq n - (n-1) = 1$$

and so $\mu_n \in \sigma(H)$. But $\dim P_{(\mu_n-\varepsilon_0,\, \mu_n+\varepsilon_0)}$ $< \dim P_{(-\infty,\, \mu_n+\varepsilon_0)} < \infty$, so $\mu_n \in \sigma_{disc}(H)$. Therefore μ_n is an eigenvalue and we can find $\delta > 0$ such that $(\mu_n - \delta, \mu_n + \delta) \cap \sigma(H) = \{\mu_n\}$. Then $\dim P_{(-\infty,\, \mu_n]}$ $= \dim P_{(-\infty,\, \mu_n+\delta)} \geq n$ by (156.2) so there are at least n eigenvalues $E_1 \leq \cdots \leq E_n \leq \mu_n$. If $E_n < \mu_n$, then $\dim\left(P_{-\infty,\, \frac{E_n+\mu_n}{2}}\right) \geq n$ contradicting (156.1). Thus $E_n = \mu_n$ i.e. μ_n is the n^{th} eigenvalue. This shows we are in situation (a). As either case I or case II must hold, we must have $\mu_n(H) \leq \inf\{\lambda : \lambda \in \sigma_{ess}(H)\}$, proving (154.5). Also if $\mu_n < \inf\{\lambda : \lambda \in \sigma_{ess}(H)\}$, then case I cannot hold, so II holds, and hence (a) is verified. On the other hand, if $\mu_n = \inf\{\lambda : \lambda \in \sigma_{ess}(H)\}$, then case II cannot hold, so I holds, and hence (b) is verified. $\square$

There is another formulation of the min-max principle which differs on a technical point and which is important and very useful:

Th^m 160.1

If H is s.adj and bdd below, then

(160.1) $$\mu_n = \sup_{\varphi_1, \dots, \varphi_{n-1}} \inf_{\substack{\varphi \perp \langle \varphi_1, \dots, \varphi_{n-1} \rangle \\ \varphi \in Q(H),\ \|\varphi\| = 1}} (\varphi, H\varphi)$$

where $Q(H)$ is the form domain of H.

Proof: Let $\tilde{\mu}_n$ be the RHS of (160.1). Clearly $\tilde{\mu}_n \le \mu_n$ as $D(H) \subset Q(H)$. Show by construction that we can always modify $\tilde{\varphi} \in Q(H)$ to a $\varphi \in D(H)$ appropriately to prove that $\tilde{\mu}_n = \mu_n$. $\square$

Remark: In the proof of Theorem 154.2, note that it is immaterial if $\varphi \in D(H)$ or $Q(H)$ in the calculations and so $\tilde{\mu}_n$ is necessarily the same as μ_n. Check this!

Applications of min-max: Note

(1) min-max ideal for comparing eigenvalues of operators

(2) useful in locating where σ_{ess} begins

(3) If we know that $\sigma_{ess}(H) = [a, \infty)$ and in

addition, we know $\mu_n < a$. Then we can conclude immediately that H has at least n eigenvalues!

First we prove some matrix results.

Theorem 161.1

Let H be an $n \times n$ Hermitian matrix and let $V = a\,(u_0, \cdot)\,u_0$ be rank 1, $a \in \mathbb{R}$, $\|u_0\| = 1$. Then the eigenvalues of H and $H+V$ interlace

i.e. if $a > 0$,

$$\lambda_1(H) \le \lambda_1(H+V) \le \lambda_2(H) \le \lambda_2(H+V) \le \cdots$$

and if $a < 0$

$$\lambda_1(H+V) \le \lambda_1(H) \le \lambda_2(H+V) \le \lambda_2(H).$$

Proof: Suppose $a > 0$. The case $a < 0$ follows by interchanging the roles of H and $H+V$. By min-max we clearly have

$$\mu_j(H) \le \mu_j(H+V) \qquad 1 \le j \le n.$$

Fix $u_1, \ldots, u_{j-1}$. Then

$$U_{H+V}(u_1, \cdots, u_{j-1}) = \inf_{\substack{u \perp \langle u_1, \cdots, u_{j-1} \rangle \\ \|u\|=1}} (u, H+V\, u)$$

$$= \inf_{\substack{u \perp \langle u_1, \cdots, u_{j-1} \rangle \\ \|u\|=1}} (u, Hu) + a|(u_0, u)|^2$$

$$\leq \inf_{\substack{u \perp \langle u_1, \cdots, u_{j-1}, u_0 \rangle \\ \|u\|=1}} (u, Hu) + a|(u_0, u)|^2$$

$$= \inf_{\substack{u \perp \langle u_1, \cdots, u_{j-1}, u_0 \rangle \\ \|u\|=1}} (u, Hu)$$

$$= U_H(u_1, \cdots, u_{j-1}, u_0)$$

Thus
$$\mu_j(H+V) \leq \sup_{u_1, \cdots, u_{j-1}} U_H(u_1, \cdots, u_{j-1}, u_0)$$

$$\leq \sup_{u_1, \cdots, u_{j-1}, u_j} U_H(u_1, \cdots, u_j)$$

$$= \mu_{j+1}(H). \qquad \square.$$

<u>Exercise 162.1</u>

Let H be an $n \times n$ Hermitian matrix and let $\tilde{H}$ be the $(n-1) \times (n-1)$ Hermitian matrix obtained by eliminating either the n^{th} row and n^{th} column of H, or the 1^{st} row and 1^{st} column of H. Show

That the eigenvalues of H and $\tilde{H}$ interlace.

Alternative proof of Th^m (161.1) For $z \notin$ spec H

$$\det(H+V-z) = \det\left((H+V-z)(H-z)^{-1}\right)\det(H-z)$$

$$= \det\left(I + V(H-z)^{-1}\right)\det H - z.$$

Now $\exists$ unitary U st $U^*HU = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

where $\lambda_i = \lambda_i(H)$. Thus

$$\det\left(I + V(H-z)^{-1}\right) = \det U^*\left(I + V(H-z)^{-1}\right)U$$

$$= \det\left(I + U^*VU(\Lambda - z)^{-1}\right)$$

$$= \det\left(I + a[(v_0, \cdot)v_0](\Lambda - z)^{-1}\right)$$

where $v_0 = U^*u_0$. But if $T = I + (\tilde{u}, \cdot)\tilde{v}$

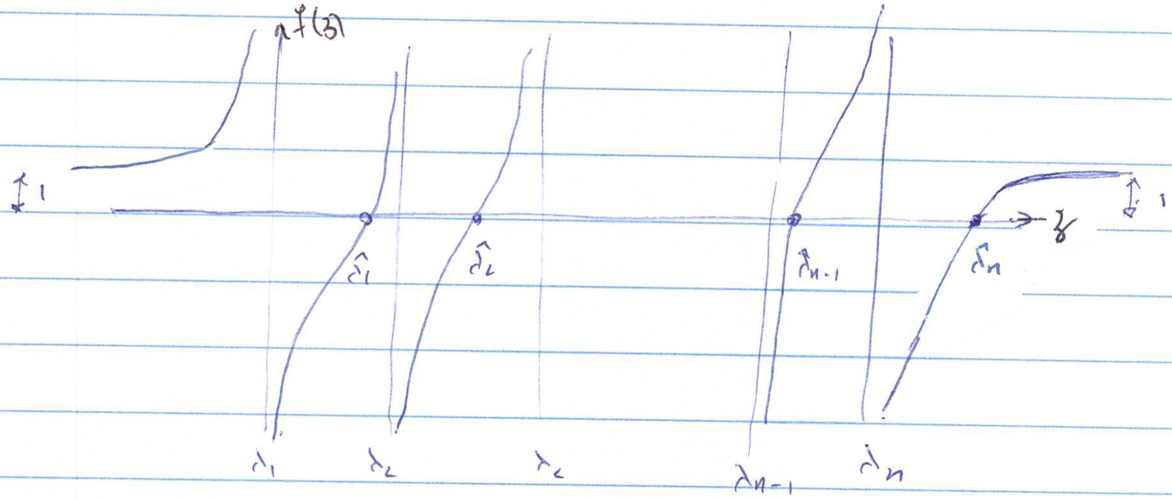
then $\det T = 1 + (\tilde{u}, \tilde{v})$ (why?) and so

$$\det\left(I + V(H-z)^{-1}\right) = 1 + a\left(v_0, (\Lambda - z)^{-1}v_0\right)$$

$$= 1 + a\sum_{i=1}^{n}\frac{|v_{0,i}|^2}{\lambda_i - z} \equiv f(z)$$

By genericity, (exer. below) $\lambda_i \neq \lambda_j$ for $i \neq j$ and $|v_{0,i}|^2 > 0$ $\forall i$. (The general result follows by continuity. Exercise!)

Then for real $z \notin$ spec H, $f(z)$ has the form



There are clearly n real points $\hat\lambda_j$ st $f(\hat\lambda_j) = 0$.

These are the n interlacing eig's of $H+V$.

(***) Exercise: Show that $S = \{H : \sigma(H)$ is simple and $v_{0,i} \neq 0$ for all $i = 1, \ldots, n\}$ is an open dense set of full measure i.e. the measure of the complement of S in the Hermitian matrices is zero. (Equip the Hermitian matrices with Lebesgue measure i.e. $dH = \prod_i dH_{ii} \prod_{i<j} d\mathrm{Re}\,H_{ij} \prod_{i<j} d\mathrm{Im}\,H_{ij}$)

Exercise: Provide an alternative proof of exercise 162.1

Now consider the Neumann operator

$$H_{Neu} = -d^2/dx^2$$

with

$$\mathrm{Dom}(H) = \{f \in AC^2[0,1] : -f'' \in L^2, \ f'(0) = f'(1) = 0\}$$

Then H_{Neu} has purely discrete and simple spectrum

$$\{ n^2\pi^2 : n = 0, 1, 2, \dots \}$$

with a complete orthonormal basis of eigenfunctions

$$\varphi_n(x) = \sqrt{2} \cos n\pi x, \quad n > 0$$
$$= 1, \quad n = 0.$$

Necessarily, by min-max, $\mu_n(H) = n^2\pi^2$, $n = 0, 1, 2, \dots$ (why?)

Now let T be the s.adj. operator in (151.2)

with $V \in L^1(0,1)$, $T = -d^2/dx^2 + V$ on

$$D(T) = \{ f \in AC^1[0,1] : -f'' + Vf \in L^2$$
$$f'(0) = f'(1) = 0 \}$$

We would like to use min-max to conclude that the spectrum of

of T is also discrete, by comparison with H_{new}.

But there is no simple relation between the

domain of H and the domain of T. However

their form domains are identical,

$$Q(T) = Q(H) = \{ f \in AC[0,1] : f' \in L^2 \}$$

and so we can use the quadratic form version of min-max (Thm 160.1)

to make the comparison. Indeed for $\varphi \in Q(T)$

$= Q(H)$, we have by (151.1),

$$q_T(\varphi) = \int_0^1 |\varphi'|^2 + \int_0^1 V|\varphi|^2$$

$$\geq -\left(\int_0^1 |V(x)|dx\right)(1+\varepsilon^{-1})\int_0^1 |\varphi|^2$$

$$+ (1-\varepsilon)\int_0^1 |\varphi'|^2$$

$$\geq (1-\varepsilon)\, q_H(\varphi) - M\|\varphi\|^2 ,$$

where $M = \left(\int_0^1 |V(x)|dx\right)(1+\varepsilon^{-1})$

Hence for $0<\varepsilon<1$, by min-max,

$$\mu_n(T) \geq (1-\varepsilon)\mu_n(H) - M$$

$$= (1-\varepsilon)\, n^2\pi^2 (H) - M$$

It follows that $\mu_n(T) \to \infty$ as $n \to \infty$

and hence case (b) in Thm 154.2 does not

arise. Thus T has discrete spectrum and each

$\mu_n(T)$ is an eigenvalue (in fact the n^{th}, counting multipl.).

Exer: Note that the eigenvalues of T are simple. Why?

Lecture 13 We now show how to use min-max to prove Weyl's famous result that for $-\Delta_D^\Omega$ with Dirichlet boundary conditions in a bounded region $\Omega \subset \mathbb{R}^m$, $N(\lambda) = \dim P_{(-\infty,\lambda)}(-\Delta)$ is asymptotically equal to $C_m \lambda^{m/2}$ times the volume of Ω where

$$C_m = \left[m\, 2^{m-1} \pi^{m/2}\, \Gamma\left(\tfrac{m}{2}\right) \right]^{-1}$$

where Γ is the Euler gamma function. Noting that $C_m = \tau_m / (2\pi)^m$ where τ_m is the volume of the unit ball in $\mathbb{R}^m$, we can write Weyl's result in the form

(166.1) $$N(\lambda) \sim \tau_m \lambda^{m/2} (\operatorname{vol} \Omega)(2\pi)^{-m}$$

In fact since $\tau_m \lambda^{m/2} = \operatorname{vol}\{p \in \mathbb{R}^m : p^2 < \lambda\}$ we see that

(166.2) $$N(\lambda) \sim \operatorname{vol}\{\langle x, p\rangle \in \mathbb{R}^{2m} : x \in \Omega,\ p^2 < \lambda\} / (2\pi)^m$$