(133.1) $q(\varphi, \psi) = (\varphi, A\psi) \quad \forall \varphi \in Q(q)$.

Uniqueness of A follows from the following:

Lecture 11: We now describe $D(A)$ in a more concrete fashion.

We have:

(133.2) Claim: $D(A) = \{ \varphi \in Q(q) : q(\psi, \varphi) = (\psi, x_\varphi)$ for some x_φ an $\forall \psi \in Q(q) \}$

Furthermore, if $\varphi \in D(A)$, then $A\varphi = x_\varphi$

Proof: From (131.1) we have for $\varphi, \psi \in D(B) = D(A)$

(133.3) $$q(\psi, \varphi) = (\psi, A\varphi)$$

But $D(A)$ is $\|\cdot\|_{+1}$ - dense in $\mathcal{H}_{+1}$ hence (133.3) holds $\forall$ $\psi \in Q(q)$ with $x_\varphi = A\varphi$. Thus $D(A) \subset$ RHS of the Claim

Conversely suppose for some x_φ

$$q(\psi, \varphi) = (\psi, x_\varphi) \qquad \forall \psi \in Q(q)$$

ie.

$$(\psi, \varphi)_{+1} = (\psi, x_\varphi) + (\psi, \varphi) = (\psi, x_\varphi + \varphi)$$

But $(\psi, \varphi)_{+1} = [\hat{B}\varphi](\psi) \quad \therefore \quad [\hat{B}\varphi](\psi) = (\psi, x_\varphi + \varphi) = [j(x_\varphi + \varphi)](\psi)$

$\forall \psi \in Q(q)$. Hence $\varphi = \hat{B}^{-1} \circ j\,(x_\varphi + \varphi) \in D(B) = D(A)$. Thus RHS $\subset D(A)$

Also $B\varphi = j^{-1} \hat{B}\, \hat{B}^{-1} \circ j\,(x_\varphi + \varphi) = x_\varphi + \varphi$ ie $A\varphi = x_\varphi$.

~~Thus:~~ $\cancel{(\psi, \varphi)_{+1} = (\psi, B\varphi)}$ ~~or~~

$\cancel{q(\psi, \varphi) = (\psi, A\varphi) \quad \forall \psi \in \mathcal{H}_{+1}}$.

This proves the claim.

Remark Clearly it is enough to know that $q(\psi, \varphi) = (\psi, x_\varphi)$ $\forall \psi$ in a core for q, to conclude that $q(\psi, \varphi) = (\psi, x_\varphi)$ $\forall \psi \in Q(q)$

A is uniquely determined by (133.2) in the following sense. Let $\tilde{A}$ be a self-adjoint operator with $D(\tilde{A}) \subset Q(q)$ and for every $\psi \in D(\tilde{A})$

(134.1) $$q(\varphi, \psi) = (\varphi, \tilde{A}\psi)$$

$\forall \varphi$ belonging to a core for q. Taking limits in (134.1), we have

$$q(\varphi, \psi) = (\varphi, \tilde{A}\psi)$$

$\forall \varphi \in Q(q)$. But then by (133.2), $\psi \in D(A)$ and

$$A\psi = \tilde{A}\psi$$

Thus $\tilde{A} \subset A$ and so $\tilde{A} = A$ as A and $\tilde{A}$ are s-adjoint

This completes the proof of Theorem 129.1

Note the following: A symm. oper always has a closed symmetric extension: but it may be that none of its closed extensions is s.adj. On the other hand, semi-bded forms need not have any closed extensions, but when such extensions exist, they are the quad. forms associated with s.adj operators

Caveats:

(1) As we have seen on many occasions, if A and B are self-adjoint, and $D(A) \subset D(B)$ with $B \restriction D(A) = A$, then $A = B$. But it can happen that a and b are closed semibounded quadratic forms and $b \restriction Q(a) \times Q(a) = a$, but without having $a = b$

(2) Let A be a symmetric oper. that is semi-bded. Let q be a quad. form $q(\varphi, \psi) = (\varphi, A\psi)$ with $Q(q) = D(A)$. Suppose q has a closure (it always will: see later!) $\hat{q}$, that is the smallest closed form that extends it. Then the s.adj. oper. $\hat{A}$ corres. to $\hat{q}$ may be bigger than the operator closure of A.

(3) As noted above, Quad. forms that come directly from semi-bded operators always have closures & hence semi-bded operators always have s.adj extensions; a fact that we already knew from Corollary 100.1, where we saw that $n_+ = n_-$.

The following examples illustrate (1)(2) above.

Recall the operators T_0, $T_{a,b}$, $T_{\infty,\infty}$ defined earlier as $-d^2/dx^2$ acting on

$$D_0 = \{ f \in AC^2[0,1] : f'' \in L^2,\ f(0) = f(1) = 0 = f'(0) = f'(1) \}$$

$$D_{a,b} = \{ f \in AC^2[0,1] : f'' \in L^2,\ a f(0) + f'(0) = 0,\ b f(1) + f'(1) = 0 \}$$

$$D_{\infty,\infty} = \{ f \in AC^2[0,1] : f'' \in L^2,\ f(0) = f(1) = 0 \}$$

respect. T_0 is closed and symmetric, but not s.adj. $T_{a,b}$ and $T_{\infty,\infty}$ are s.adj.

(i) Then if $q_0(\varphi, \psi) = (\varphi, T_0 \psi)$ for $\varphi, \psi \in D_0$, the form q_0 has a smallest closed extension $\hat{q}_0 = q_{\infty,\infty}$.

$$Q(q_{\infty,\infty}) = \{ f \in AC[0,1] : f' \in L^2,\ f(0) = f(1) = 0 \}$$

This extension is the form associated with $T_{\infty,\infty}$, which illustrates the remark (2) above.

(ii) The form $t_{a,b}$ associated with $T_{a,b}$ (see details below) has a form domain $Q(t_{a,b})$ which strictly contains the form $Q(q_{\infty,\infty})$ of $q_{\infty,\infty}$, the form associated with $T_{\infty,\infty}$, and $q_{a,b} \restriction Q(q_{\infty,\infty}) = q_{\infty,\infty}$. This illustrates remark (1) above.

Let

$$Q(q_{a,b}) = \{ \varphi \in AC[0,1] : \varphi' \in L^2(0,1) \}$$

(137.0) $$q_{a,b}(\varphi,\psi) = \int_0^1 \bar{\varphi}'\psi'\,dx - a\,\overline{\varphi(0)}\,\psi(0) + b\,\overline{\varphi(1)}\,\psi(1)$$

for $\varphi, \psi \in Q(q_{a,b})$

To see that $q_{a,b}$ is <u>closed</u> on $Q(q_{a,b})$ note that

for $\varphi \in Q(q_{a,b})$

$$\varphi^2(x) = \varphi^2(y) + 2\int_y^x \varphi'(t)\,\varphi(t)\,dt \quad , 0 \le x, y \le 1,$$

and so for any $\varepsilon > 0$

$$|\varphi(x)|^2 \le |\varphi(y)|^2 + \varepsilon \int_0^1 |\varphi'|^2 + \varepsilon^{-1} \int_0^1 |\varphi|^2$$

and then integrating wrt y we find

(137.1) $$|\varphi(x)|^2 \le (1+\varepsilon^{-1}) \int_0^1 |\varphi|^2\,dt + \varepsilon \int_0^1 |\varphi'|^2\,dt$$

Setting $x = 0$ or 1 in (137.1), it is easy to see (exercise) that $q_{a,b}$ is closed on $Q(q_{a,b})$, and semi-bdd.

Now let $\psi \in D(A)$, where A is the s.adj operator assoc. with $q_{a,b}$ by Thm 129.1.

Then $\forall\ \varphi \in Q(q_{a,b})$

(138.1) $$q(\varphi, \psi) = (\varphi, h)$$

for some $h \in L^2$ and $h = A\psi$

In particular (138.1) is true $\forall\ \varphi \in C_0^\infty(0,1)$

For such φ, $q(\varphi,\psi) = \int_0^1 \bar{\varphi}'\psi'$ and

$(\varphi, h) = \int \bar{\varphi} h = -\int \bar{\varphi}' H$ where $H(x) = \int_0^x h(t)\,dt$

and so $\int \bar{\varphi}'(\psi' + H) = 0 \quad \forall\ \varphi \in C_0^\infty(0,1)$.

As before, this $\Rightarrow$ $\psi' + H =$ constant. Hence,

$\psi \in AC^2(0,1)$ and $\psi'' = -H' = -h \in L^2$

Thus $D(A) \subset \underbrace{AC^2(0,1) \cap \{\psi'' \in L^2\}}_{\cap\, Q(q_{a,b})}$, $A\psi = -\psi''$ for $\psi \in D(A)$.

Now for $\psi \in D(A)$, we have for any $\varphi \in Q(q_{a,b})$

$$q(\varphi,\psi) = \int_0^1 \bar{\varphi}'\psi' - a\,\overline{\varphi(0)}\,\psi(0) + b\,\overline{\varphi(1)}\,\psi(1)$$

$$= (\varphi, A\psi) = -\int_0^1 \bar{\varphi}\,\psi''$$

Integrating RHS by parts we find

$$-\int_0^1 \bar{\varphi}\, \psi'' = +\int_0^1 \bar{\varphi}'\, \psi' - \overline{\varphi(1)}\, \psi'(1) + \overline{\varphi(0)} \cdot \psi'(0)$$

Now for $\varphi \in Q(q_{a,b})$ st $\varphi(1) = 0$, $\varphi(0) \neq 0$.

we see that we must have

$$\overline{\varphi(0)}\, \psi'(0) = -a\, \overline{\varphi(0)}\, \psi(0)$$

and so

(139.1) $$\psi'(0) + a\, \psi(0) = 0$$

Similarly from $\varphi(0) \neq 0$, $\varphi(1) \neq 0$, we obtain

(139.2) $$\psi'(1) + b\, \psi(1) = 0.$$

Thus

(139.3) $$D(A) \subset \{ AC^2(0,1) : \psi'(1) + b\psi(1) = 0,\ \psi'(0) + a\, \psi(0) = 0 \}.$$

On the other hand it is easy to see (exercise) that if

if $\psi \in$ RHS of (139.3), then

$$q(\varphi, \psi) = (\varphi, -\psi'') \quad \forall \varphi \in Q(q_{a,b})$$

and so $\psi \in D(A)$. Thus

$$A\psi = -\psi'' \quad \text{with domain}$$

$$D(A) = \{ AC^2[0,1] : \psi'(1) + b\psi(1) = \psi'(0) + a\, \psi(0) = 0 \}.$$

As already noted, it follows from Corollary 100.1, that a semi-bded symmetric operator A, $(\varphi, A\varphi) \geq -M\|\varphi\|^2$, has equal deficiency indices and hence such an operator always has s.adj. extensions. It turns out that there is a distinguished extension of A, called the *Friedrichs extension*, which is obtained from the quad. form assoc. with A.

Thm 140.1 (the Friedrichs extension)

Let A be a positive symmetric operator and let

(140.1) $$q(\varphi, \psi) = (\varphi, A\psi)$$

for $\varphi, \psi \in D(A)$. Then q is a closable quad. form and its closure $\hat{q}$ is the quad. form of a unique s.adj oper. $\hat{A}$. $\hat{A}$ is a pos. extension of A, and the lower bound of its spectrum is the lower bound of q. Further, $\hat{A}$ is the only s.adj extension of A whose domain is contained in the form domain of $\hat{q}$.

Notation: $q(u) = q(u,u)$, $u \in Q(q)$.

a closed extension.

Defn 141.0 A semi-bounded form q is closable ~~(i.e. has a closed extension)~~ iff $u_n \to 0$, $q(u_n - u_m) \to 0$ as $n,m \to \infty$ $\Rightarrow$ $q(u_n) \to 0$

Lemma 141.2

A semi-bdd form q is closable, if and only if q has a closure (the smallest closed extension) $\hat{q}$ defined in the following way. $Q(\hat{q})$ is the set of all $u \in \mathcal{H}$ st $\exists$ a seq. $u_n \in Q(q)$ st

$$u_n \to u, \quad q(u_n - u_m) \to 0$$

and for $u, v \in Q(\hat{q})$

$$\hat{q}(u,v) = \lim_{n\to\infty} q(u_n, v_n) \qquad (141.2)$$

for any $u_n \to u$, $q(u_n - u_m) \to 0$ and $v_n \to v$, $q(v_n - v_m) \to 0$, $u_n, v_n \in Q(q)$

Remark: We have seen (see p128) that in general a semi-bdd quad form may not have a closed extension.

Proof: Let q_1 be a closed ext. of q. Then if $u_n \to 0$, $q(u_n - u_m) \to 0$, $u_n \in Q(q)$, then

$$u_n \to 0, \quad q_1(u_n - u_m) = q(u_n - u_m) \to 0$$

and so by the closedness of q_1

$$q(u_n) = q_1(u_n) = q_1(u_n - 0) \to 0$$

This proves necessity.

Wlog assume $q \geq 0$ (why?). Hence we have Cauchy Schwartz,

$|q(u,v)| \le q^{\frac12}(u)\, q^{\frac12}(v)$ and the triangle ineq., $q^{\frac12}(u+v) \le q^{\frac12}(u) + q^{\frac12}(v)$

Conversely, to prove sufficiency, let $Q(\hat{q})$ be defined as in the Lemma. Then for u_n, v_n as above

$$|q(u_n, v_n) - q(u_m, v_m)|$$

$$\le |q(u_n - u_m, v_n)| + |q(u_m, (v_n - v_m))|$$

$$\le |q(u_n - u_m; u_n - u_m)|^{\frac12}\, |q(v_n, v_n)|^{\frac12} + (q(u_m, u_m))^{\frac12}\, q^{\frac12}(v_n - v_m, v_n - v_m)$$

~~Here we have assumed that $q(u,u) \ge 0$; if $q(u,u) \ge -m(u,u)$, just shift $q \to q + m$~~

Now $q(u_n - u_m, u_n - u_m) \to 0$ and as

$$q(v_n, v_n)^{\frac12} \le (q(v_n - v_m, v_n - v_m))^{\frac12} + (q(v_m, v_m))^{\frac12}.$$

we see that $q(v_n, v_n)$ is bounded. Similarly for $q(v_n - v_m, v_n - v_m)$ and $q(u_m, u_m)$. It follows that $\{q(u_n, v_n)\}$ is Cauchy and hence

(142.1) $$\hat{q}(u,v) = \lim_{n\to\infty} q(u_n, v_n) \quad \exists .$$

The limit in (142.1) is independent of the seq's $\{u_n\}, \{v_n\}$. Indeed, let $\{u_n'\}, \{v_n'\}$ be other sequences ST $u_n' \to u$, $q(u_n'-u_m', u_n'-u_m') \to 0$ and $v_n' \to v$, $q(v_n'-v_m', v_n'-v_m') \to 0$

Then $u_n'-u_n \to 0$ and

$$\left(q((u_n'-u_n)-(u_m'-u_m), (u_n'-u_n)-(u_m'-u_m))\right)^{\frac{1}{2}}$$

$$\leq q^{\frac{1}{2}}(u_n'-u_m', u_n'-u_m') + q^{\frac{1}{2}}(u_n-u_m, u_n-u_m)$$

$\to 0$ as $n, m \to \infty$.

and hence, by the assumption on q,

$$q(u_n'-u_n, u_n'-u_n) \to 0.$$

and similarly for $v_n'-v_n$.

Hence $q(u_n', v_n') = q(u_n'-u_n, v_n') + q(u_n, v_n'-v_n) + q(u_n, v_n) = q(u_n, v_n) + o(1)$

This proves that $\hat{q}$ is well-defined by (142.1).

We note that

(143.1) $$\hat{q}(u_n-u) \to 0 \quad \text{if } u_n \to u, \ q(u_n-u_m) \to 0.$$

In fact by (141.2) applied to u_n-u,

$$\lim_{n\to\infty} \hat{q}(u_n-u) = \lim_{n\to\infty}\lim_{m\to\infty} \hat{q}(u_n-u_m) = 0$$

Clearly, by construction, $\hat{q}$ is semi-bdd and $\hat{q}$ is an extension of q. We show that $\hat{q}$ is closed.

Suppose $u_n \in Q(\hat{q})$ and

$u_n \to u$ in $\mathcal{H}$ and $\hat{q}(u_n - u_m) \to 0$ as $n, m \to \infty$.

Then by (143.1), for each n, $\exists\, v_n \in Q(q)$

s.t. $\|v_n - u_n\| \le \frac{1}{n}$, $\hat{q}^{\frac{1}{2}}(v_n - u_n) \le \frac{1}{n}$.

Hence $\|v_n - u\| \le \frac{1}{n} + \|u_n - u\| \to 0$

and $q(v_n - v_m) = \hat{q}^{\frac{1}{2}}(v_n - v_m) \le \hat{q}^{\frac{1}{2}}(v_n - u_n) + \hat{q}^{\frac{1}{2}}(v_m - u_m)$

$+ \hat{q}^{\frac{1}{2}}(u_n - u_m)$.

$\le \frac{1}{n} + \frac{1}{m} + \hat{q}^{\frac{1}{2}}(u_n - u_m)$

$\to 0$ as $n, m \to \infty$

Hence $\hat{q}$ is closed. Clearly $\hat{q}$ is the smallest closed extension of q (why?)

<u>Proof of Th^m 140.1</u>

We must show that q is closable on $D(A)$

ie $u_n \to 0$, $q(u_n - u_m) \to 0$, $u_n \in D(A) \Rightarrow q(u_n) \to 0$.

But

$$q(u_n) = q(u_n, u_n)$$

$$= q(u_n, u_n - u_m) + q(u_n, u_m)$$

Thus $$|q(u_n)| \leq q^{\frac{1}{2}}(u_n)\, q^{\frac{1}{2}}(u_n - u_m) + |q(u_n, u_m)|$$

$$= q^{\frac{1}{2}}(u_n)\, q^{\frac{1}{2}}(u_n - u_m) + |(Au_n, u_m)|$$

$$\leq q^{\frac{1}{2}}(u_n)\, q^{\frac{1}{2}}(u_n - u_m) + \|Au_n\|\,\|u_m\|$$

Let $\varepsilon > 0$ be given.

Now given $\varepsilon > 0$ $\exists$ N such that for $n, m \geq N$,

$$q^{\frac{1}{2}}(u_n - u_m) \leq \varepsilon.$$

Hence for $m \geq n \geq N$, as $q^{1/2}(u_n)$ is bded, say $q^{1/2}(u_n) \leq c$,

$$|q(u_n)| \leq c\,\varepsilon + \|Au_n\|\,\|u_m\|$$

Letting $m \to \infty$ we see that $|q(u_n)| \leq c\,\varepsilon$ i.e.

$q(u_n) \to 0$. Thus q is closable, and hence has a closed extension. We conclude that there exists a s.adj.

operator $\hat{A}$ with $D(\hat{A}) \subset Q(\hat{q})$, Moreover

Lecture 10

$\psi \in D(\hat{A})$ iff

$$\hat{q}(\varphi, \psi) = (\varphi, h) \qquad \forall \varphi \in Q(\hat{q}).$$

and then

$$\hat{A}\psi = h.$$

We now show that $\hat{A}$ is an extension of A

If $\varphi, g \in D(A)$ then by construction

$$(A\varphi, g) = q(\varphi, g)$$

and hence

$$(A\varphi, \psi) = \hat{q}(\varphi, \psi)$$

for any $\psi \in Q(\hat{q})$ as $D(A)$ is a core for $\hat{q}$ (why?). In particular for

$\psi \in D(\hat{A})$, $\quad (A\varphi, \psi) = \hat{q}(\varphi, \psi) = (\varphi, \hat{A}\psi)$

Hence $\varphi \in D(\hat{A}^*) = D(\hat{A})$ & $\hat{A}^*\varphi = \hat{A}\varphi = A\varphi$

So $\hat{A}$ extends A, as claimed above.

Insert 146+

The statement about the spectrum of $\hat{A}$ is left as an

easy exercise. $\square$

We now show that $\hat{A}$ is the unique s.adj extension of A with $D(\hat{A}) \subset Q(\hat{q})$. Indeed, suppose $\tilde{A}$ is a s.adj. ext. of A with $D(\tilde{A}) \subset Q(\hat{q})$.

For $\psi \in D(\tilde{A})$ and $\varphi \in D(A) \subset D(\tilde{A})$, have

$$(\varphi, \tilde{A}\psi) = (\tilde{A}\varphi, \psi) = (A\varphi, \psi)$$

Now for any $f, g \in D(A) \subset Q(\hat{q})$.

$$\hat{q}(f,g) = (Af, g)$$

and as $D(A)$ is a core for $Q(\hat{q})$ we have

$$\hat{q}(f,g) = (Af, g) \qquad \forall f \in D(A),\ g \in Q(\hat{q}).$$

In particular, for $f = \varphi$ and $g = \psi \in D(\tilde{A}) \subset Q(\hat{q})$

we have $(\varphi, \tilde{A}\psi) = (A\varphi, \psi) = \hat{q}(\varphi, \psi)$, $\forall \varphi \in D(A)$

Again noting that $D(A)$ is a core for $\hat{q}$, we conclude that $\hat{q}(\varphi,\psi) = (\varphi, \tilde{A}\psi)$ $\forall \varphi \in Q(\hat{q})$

But then by (133.2), $\psi \in D(\hat{A})$ and $\hat{A}\psi = \tilde{A}\psi$

Thus $\tilde{A} \subset \hat{A}$ and as both are s.adj. op's, $\hat{A} = \tilde{A}$.

Example

Let $A = -d^2/dx^2$ with domain $D(A) = C_0^\infty(0,1)$.

in $\mathcal{H} = L^2(0,1)$. Then

$$q(u,v) = \int_0^1 \bar{u}\, Av = \int_0^1 \bar{u}(-v'')$$

$$= \int_0^1 \bar{u}' v' \quad , \quad u, v \in D(A)$$

& so $q \geq 0$. Hence q is closable and (exercise)

$$\hat{q}(u) = \int_0^1 |u'|^2 dx \, , \qquad Q(\hat{q}) = \{ u \in AC[0,1] : u' \in L^2 \\ u(0) = u(1) = 0 \}$$

Also the Friedrichs extension H of A has domain (exer)

$$D(H) = \{ f \in L^2 : f \in AC^2[0,1], \; f'' \in L^2, \\ f(0) = f(1) = 0 \}$$

$$Hf = -f'' \quad \text{for} \quad f \in D(H)$$

Thus H is the Dirichlet operator.

If we take

$$\tilde{q}(u,v) = \int_0^1 \bar{u}' v' \quad , \quad \text{for } u, v \in C^\infty[0,1].$$

then $\tilde{q}(u,v) = \int_0^1 \bar{u}(-v'') = \int_0^1 \bar{u}(Av)$

when $\operatorname{dom} A = \{ f \in C^\infty[0,1] : f'(0) = f'(1) = 0 \}$, $Af = -f''$.

and the Friedrichs extension of $\dot{A}$ is the Neumann Operator.

<u>Example 1</u> (Weak solutions of PDE's).

Let Ω be an open region in $\mathbb{R}^n$ and let A be the operator $-\Delta + I$ with domain $C_0^\infty(\Omega) \subset L^2(\Omega)$. A is symmetric and bdd below by 1. If $\hat{A}$ is the Friedrichs extension of A, then $\hat{A} \geq I$, & so Ran $\hat{A} = L^2(\Omega)$. Thus for any $g \in L^2(\Omega)$ $\exists$ $f \in D(\hat{A})$ so that $\hat{A} f = g$.

$D(A) = C_0^\infty(\Omega)$ is a core for $\hat{q}$

Now using 1, $\exists$ $f_n \in C_0^\infty(\Omega)$, $f_n \to f$ in $\|\cdot\|_{+1}$, so that for any $u \in C_0^\infty(\Omega) = D(A)$, $\hat{q}(u, f) = \lim_{n\to\infty} \hat{q}(u, f_n) = \lim_{n\to\infty} q(u, f_n)$

$$= \lim_n \int u(-\Delta+1) f_n = \lim_n \int [(-\Delta+1)u] f_n = \int (-\Delta+1)u \, f$$

But $\hat{q}(u, f) = (u, \hat{A} f) = (u, g)$. Thus for any $g \in L^2$, the equation $(-\Delta+1) f = g$ has a weak solution in $L^2(\Omega)$. Since $\Delta f = f - g \in L^2(\Omega)$ we can use Sobolev theory to show a certain amount of regularity for f. If $g \in C_0^\infty(\Omega)$, then applying Δ repeatedly to the eqtn $\Delta f = f - g$ proves $f \in \bigcap_{m=1}^{\infty} W_m(\Omega)$, so in this case Sobolev Th^y $\Rightarrow$ f is C^∞.

Example 2

Let A be a closed, densely defined operator and let
$$D(A^*A) = \{\psi \in D(A) : A\psi \in D(A^*)\}$$
Then A^*A on $D(A^*A)$ with $A^*A\psi \equiv A^*(A\psi)$ is s.adj.

Remark: It is not evident a priori that $D(A^*A)$ contains any vectors other than the zero vector.

Proof of Exple 2 $\quad q(\psi, \varphi) = (A\psi, A\varphi)$ is a positive quad. form on $D(A) \times D(A)$. Also as A is a closed operator, q is a closed form.

Hence $\exists$ a s.adj. operator T with

$D(T) \subset Q(q) = D(A)$ such that $\psi \in D(T) \iff$

$q(\varphi, \psi) = (\varphi, h)$, $\forall \varphi \in D(A)$ for some

$h \in \mathcal{H}$. In this case, $h = T\psi$.

But if $q(\varphi, \psi) = (A\varphi, A\psi) = (\varphi, h)$. $\forall \varphi \in D(A)$

then $A\psi \in D(A^*)$ and $A^*A\psi = h = T\psi$.

Hence $D(T) \subset D(A^*A)$ and $T\psi = A^*(A\psi)$
if $\psi \in D(T)$.

Conversely if $\psi \in D(A^*A)$ then $A\psi \in D(A^*)$

and $q(\varphi, \psi) = (A\varphi, A\psi) = (\varphi, A^*A\psi)$ $\forall \varphi \in D(A)$

Thus $\psi \in \mathrm{Dom}\, T$ and $T\psi = A^*A\psi$ Thus $D(A^*A) \subset$

$D(T) \Rightarrow \mathrm{Dom}\, T = D(A^*A)$ and $T\psi = A^*A\psi$ on $D(T)$.

Lecture 12

Exple 3

Let $\mathcal{H} = L^2(0,1)$ and let $V(x) \in L^1(dx)$, V real valued.