Special Topics in Analysis : Spring 2016

Lecture 1

This course is a continuation of the Functional Analysis course that I gave last semester. The goal of this course is

(1.1) to develop the general theory of bounded and unbounded operators in Hilbert Space

(1.2) to develop the spectral theory of self-adjoint operators in Hilbert space

(1.3) to analyze in detail the spectral theory of some concrete self-adjoint operators, particularly Schrödinger operators in quantum mechanics. Here we will be analyzing point spectrum vs continuous spectrum, and also scattering theory.

References

[RS] Reed-Simon, Methods of Modern Mathematical Physics, Volumes I, II, III and IV.

Before we start on (1.1)(1.2)(1.3), we will first consider a problem left over from last semester. Although we considered the general theory of Fredholm operators, and in particular compact operators, we did not analyze a concrete problem arising in the theory from mathematical physiss. So we will consider the following problem:

Let Ω be an open, bounded connected set in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Given a function $f \in C(\partial\Omega)$, we would like to solve

$$(2.1) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

for a $C^2(\Omega)$ function $u = u(x)$ which is continuous on $\bar{\Omega}$, We will follow the presentation of the solution by Jim Portegies, who took a similar course from me in Spring 2011 while he was a student at Courant. Portegies presentation is based on RS I, but with more details.

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Exercise 1 problem set 7

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In this exercise, we solve the Dirichlet problem by means of a double layer potential method and the compact operator theorem. We assume that $\Omega \subset \mathbb{R}^3$ is a connected open set with smooth boundary $\partial\Omega$. By the latter, we mean that for every $x_0 \in \partial\Omega$, there exists an $r_{x_0} > 0$ and a C^∞ function $g^{(x_0)} : \mathbb{R}^2 \to \mathbb{R}$, $g^{(x_0)}(0) = 0$, $\nabla g^{(x_0)}(0) = 0$, such that, upon rotation and translation (without loss of generality assuming $x_0 = 0$),

$$\Omega \cap B_{r_{x_0}}(x_0) = \{z \in B_{r_{x_0}}(x_0) \mid z_3 > g(z_1, z_2)\}, \tag{1}$$

where we use the notation $B_r(a)$ for an open ball with radius r around a point a, that is

$$B_r(a) := \{x \in \mathbb{R}^3 \mid |x - a| < r\},$$

see Fig. 1.

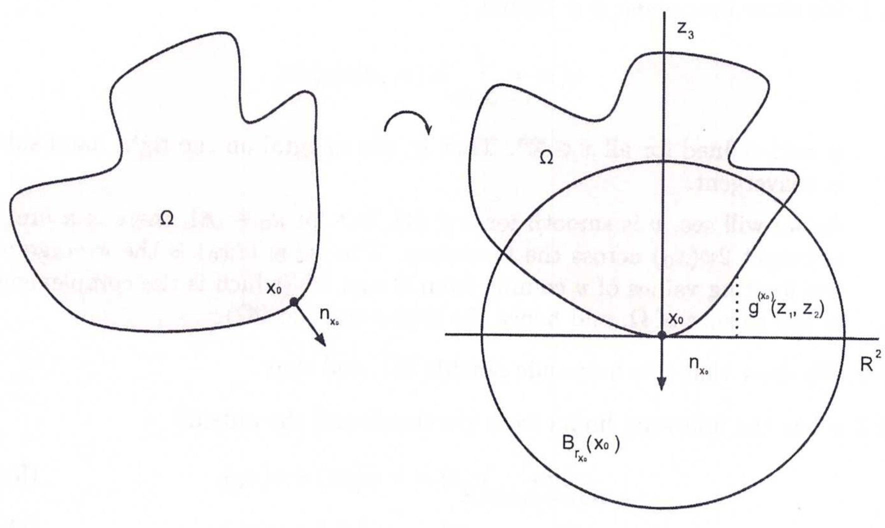


Figure 1: After rotating and translating, the surface $\partial\Omega$ can locally around x_0 be written as a graph of a function $g^{(x_0)}$, according to (1).

Given a function $f \in C(\partial\Omega)$, we would like to solve

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

For $y \in \partial\Omega$, we define

$$K(x,y) = \frac{1}{2\pi}\frac{(x-y,n_y)}{|x-y|^3},$$

where n_y denotes the exterior (unit) normal to $\partial\Omega$ at the point y. The scheme for finding a solution to the Dirichlet Problem (2) is then as follows. First, find a function $\phi \in C(\partial\Omega)$ such that

$$f(x) = -\phi(x) + \int_{\partial\Omega} K(x,y)\phi(y)dS_y, \qquad \text{for all } x \in \partial\Omega. \tag{3}$$

Then, u defined by

$$u(x) = \int_{\partial\Omega} K(x,y)\phi(y)dS_y \tag{4}$$

is harmonic and smooth on Ω, and

$$\lim_{\Omega \ni x \to x_0 \in \partial\Omega} u(x) = f(x_0).$$

Note that due to the special structure of K, the above definitions (3) and (4) are invariant under rotations and translations of the coordinate frame. Actual computations can therefore always be done locally, representing the surface by the graph of a function g.

We will show that the above scheme works. Our treatment follows Reed and Simon, volume 1, but we will work out more of the details. We divide the proof in several steps.

Step 1 We show that given $\psi \in C(\partial\Omega)$,

$$v(x) = \int_{\partial\Omega} K(x,y)\psi(y)dS_y,$$

is well-defined for all $x \in \mathbb{R}^3$. That is, the integral on the right hand side is convergent.

As we will see, v is smooth for $x \notin \partial\Omega$, but for $x_0 \in \partial\Omega$, there is a jump of height $2\psi(x_0)$ across the boundary. The value $v(x_0)$ is the average of the limiting values of v coming from Ω and $\bar{\Omega}^c$ (which is the complement of the closure of Ω, and hence the region outside $\partial\Omega$)

Step 2 We show that v is harmonic outside $\partial\Omega$, and that

Step 3 v has the following limits from the inside and the outside

$$\lim_{\Omega \ni x \to x_0 \in \partial\Omega} v(x) = -\psi(x_0) + v(x_0), \tag{5a}$$

$$\lim_{\bar{\Omega}^c \ni x \to x_0 \in \partial\Omega} v(x) = +\psi(x_0) + v(x_0). \tag{5b}$$

Step 4 Next we define the operator $T : C(\partial\Omega) \to C(\partial\Omega)$,

$$(T\psi)(x) = \int_{\partial\Omega} K(x,y)\psi(y)dS_y.$$

We need to show that T indeed maps to $C(\partial\Omega)$. At the same time, we will prove that T is compact.

Step 5 In order to show that (3) always has a solution $\phi \in C(\partial\Omega)$ for given $f \in C(\partial\Omega)$, it suffices by the compact operator theorem to show that $\mathcal{N}(-I+T) = \{0\}$. The proof requires several steps. First we realize that if $(-I+T)\phi = 0$ for some ϕ, by (5) and the maximum principle for harmonic functions, u defined by (4) is 0 on Ω. Next we show continuity of $\partial u/\partial n$ through the boundary $\partial\Omega$. By a partial integration argument, we show that u should be constant on $\bar{\Omega}^c$, and in fact 0, . By (5) we conclude that $\phi = 0$ on $\partial\Omega$. ~~and $u \equiv 0$ on $\bar{\Omega}^c$~~.

We will work out the arguments of the above points in the following sections. However, before we proceed it is useful to gather some technical results, which we will do in step 0.

Step 0

First of all, there is a constant $\tilde{C}$ such that for all $x, y \in \partial\Omega$,

$|n_x - n_y| \le \tilde{C}\,|x-y|$

$$|n_x - n_y| \cancel{|x-y|} \le \tilde{C} \cancel{|n_x - n_y|} \cdot |x-y| \tag{6}$$

To see this, we consider for each $x \in \partial\Omega$ the function $g^{(x)}$ and the radius r_x as in (1). We cover $\partial\Omega$ with the collection of open balls $\{B_{r_x/4}(x)\}_{x\in\partial\Omega}$. Since $\partial\Omega$ is compact, there is a finite subcover, corresponding to $x^1, \dots, x^N \in \mathbb{R}^3$, for some $N \in \mathbb{N}$, say. Define

$$\delta := \min_{i=1,\dots,N} r_{x^i}/4.$$

Let $|x-y| > \delta$, then

$$|n_x - n_y| \le 2 < \tfrac{2}{\delta}|x-y|.$$

If $|x-y| < \delta$, there exists an $i \in \{1,\dots,N\}$ such that $x, y \in B_{r_{x^i}}(x^i)$. As mentioned before, we can rotate and translate, so that we may assume $x^i = 0$, and $n_{x^i} = (0,0,-1)$. Then the surface can be written locally as the graph of $g^{(x^i)}$, and for $x \in B_{r_{x^i}}(x^i)$,

$$n_x = \frac{(g^{(x^i)}_{x_1}(x_1,x_2), g^{(x^i)}_{x_2}(x_1,x_2), -1)^T}{(1 + (g^{(x^i)}_{x_1}(x_1,x_2))^2 + (g^{(x^i)}_{x_2}(x_1,x_2))^2)^{1/2}}$$

where g_{x_j} means the partial derivative of g with respect to x_j, $j = 1,2$. As the right hand side is a smooth function in (x_1, x_2), we find that there is a constant C^i such that

$$|n_x - n_y| < C^i|x-y|,$$

for all $x, y \in B_{r_{x^i}}(x^i)$. If we take

$$\tilde{C} := \max(2/\delta, C^1, \dots, C^N),$$

the inequality (6) holds for all $x, y \in \partial\Omega$.

We now define the function $h : \partial\Omega \times (-1/\tilde{C}, 1/\tilde{C}) \to \mathbb{R}^3$,

$$h(x,t) = x + tn_x. \tag{7}$$

Note that h is one-to-one. Indeed, if

$$x + tn_x = y + tn_y,$$

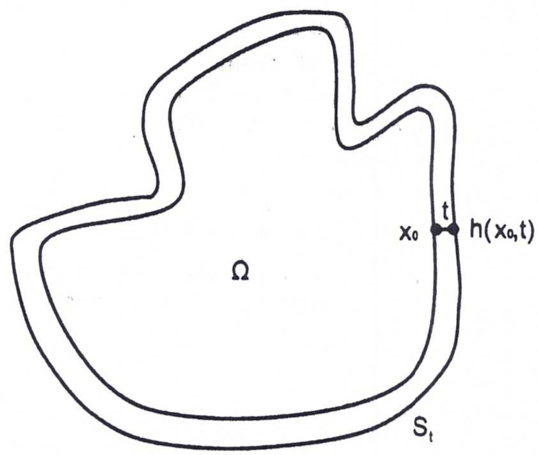


Figure 2: Illustration of the function h as defined in (7), and of the surface S_t.

$|t| < \frac{1}{\tilde{C}}$

then

$$|x-y| = |t||n_y - n_x|,$$

which, by (6) implies $x = y$ if $\cancel{|t| > 1/\tilde{C}}$. If we define $S_t := h(\partial\Omega, t)$, it follows that S_t is a smooth hypersurface as well. Moreover, it encloses a connected set, which we denote Ω_t. In other words, $S_t = \partial\Omega_t$.

Finally, choosing δ as above, it follows that for some constant $C \geq \tilde{C}$ and for any $x_0 \in \partial\Omega$, the part of the surface $B_\delta(x_0) \cap \partial\Omega$ can be represented as the graph of $g^{(x_0)}$, with uniform bounds

$$|g^{(x_0)}(y_1, y_2)| \leq C(y_1^2 + y_2^2) \tag{8a}$$

and

$$|g_{y_1}^{(x_0)}(y_1, y_2)| + |g_{y_2}^{(x_0)}(y_1, y_2)| \leq C\sqrt{y_1^2 + y_2^2}. \tag{8b}$$

1 Step 1

For $\psi \in C(\partial\Omega)$ and $x \in \mathbb{R}^3$ we consider the integral

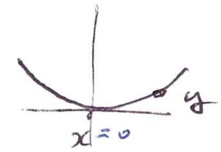
$$v(x) := \int_{\partial\Omega} K(x,y)\psi(y)dS_y. \tag{9}$$

For $x \notin \partial\Omega$, $K(x,y)$ is bounded uniformly in $y \in \partial\Omega$, and the integral on the right hand side of (9) clearly exists. To ensure existence of the integral for $x \in \partial\Omega$, let us try to estimate $|K(x,y)|$, for $x, y \in \partial\Omega$. We may by rotating and translating without loss of generality assume that $x = (0,0,0)$ and $n_x = (0,0,-1)$. In step 0 we saw that there exists a $\delta > 0$, and a smooth function $g : \mathbb{R}^2 \supset B_\delta(0) \to \mathbb{R}^3$, such that

$$\partial\Omega \cap B_\delta(0) = \{z \in B_\delta(0) | z_3 = g(z_1, z_2)\}.$$

If $|x-y| \geq \delta$,

$$|K(x,y)| = \frac{1}{2\pi}\frac{|(x-y)\cdot n_y|}{|x-y|^3} \leq \frac{1}{2\pi\delta}\frac{|x-y|^2}{|x-y|^3} = \frac{1}{2\pi\delta}\frac{1}{|x-y|}.$$



If $|x-y| < \delta$, we know that

$$\begin{aligned} |(x-y)\cdot n_y| &\leq |(x-y)\cdot(n_y-n_x)| + |(x-y)\cdot n_x| \\ &= |(x-y)\cdot(n_y-n_x)| + |g(y_1', y_2')|. \end{aligned}$$

, where $y = (y_1', y_2', g(y_1', y_2'))$ as $x = 0$

For the constant C in step 0 we know that

$$\begin{aligned} |n_y - n_x| &\leq C|y-x|, \\ |g(y_1,y_2)| &\leq C\cancel{|x-y|^2}. (|y_1'|^2 + |y_2'|^2) \leq C|y|^2 = C|x-y|^2 \end{aligned}$$

Then

$$|(x-y)\cdot n_y| \leq 2C|x-y|^2,$$

and

$$|K(x,y)| \leq \frac{1}{\pi}\frac{C}{|x-y|}.$$

It follows that there is a constant $\tilde{M}$ such that $|K(x,y)| < \tilde{M}/|x-y|$, for all $x, y \in \partial\Omega$. This singularity is integrable, so that the integral defining $v(x)$ exists. In fact, there exists a constant M such that for all $x \in \partial\Omega$, and $\gamma < \delta$,

$$\int_{\partial\Omega\cap B_\gamma(x)} |K(x,y)| dS_y \leq M\gamma. \tag{10}$$

2 Step 2

We would like to show that v defined by (9) is harmonic and smooth on $(\partial\Omega)^c$. If $x_0 \notin \partial\Omega$, there exists an $r > 0$ such that $B_{2r}(x_0) \cap \partial\Omega = \emptyset$. Then, there exists a $C_1 > 0$ such that

$$|K(x,y)|, |D_x K(x,y)|, |D_x^2 K(x,y)|, |D_x^3 K(x,y)| < C_1$$

for all $x \in B_r(x_0)$, and $y \in \partial\Omega$. Hence, for $|z| < r$, by Taylor's formula

$$\begin{aligned} &\left| v(x+z) - v(x) - \left(\int_{\partial\Omega} D_x K(x,y)\phi(y)dy\right) z + z^T\left(\int_{\partial\Omega} D_x^2 K(x,y)\psi(y)dy\right) z \right| \\ &= \left| \int_{\partial\Omega} (\cancel{v(x+z) - v(x)} - (D_x K(x,y))z - z^T(D_x^2 K(x,y))z)\psi(y)dy \right| \\ &\leq C_2|z|^3 \int_{\partial\Omega} |\psi(y)| dy, \end{aligned}$$

$\boxed{K(x+z) - K(x)}$

for some constant $C_2 > 0$, independent of z. It follows that v is twice differentiable, with

$$Dv(x) = \int_{\partial\Omega} D_x K(x,y)\psi(y)dS_y,$$

$$D^2 v(x) = \int_{\partial\Omega} D_x^2 K(x,y)\psi(y)dS_y.$$

(by direct calculation)

Because $\Delta_x K(x,y) = 0$, for $x \notin \partial\Omega$, it follows that $\Delta v(x) = 0$ for $x \notin \partial\Omega$. This also implies smoothness of v. Alternatively, smoothness of v can be proven by arguments similar to the ones that show v is twice differentiable.

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3 Step 3

We would now like to compute the boundary values of v. That is, we want to prove (5), which we restate for convenience

$$\lim_{\Omega \ni x \to x_0 \in \partial\Omega} v(x) = -\psi(x_0) + v(x_0), \tag{11}$$

$$\lim_{\bar{\Omega}^c \ni x \to x_0 \in \partial\Omega} v(x) = +\psi(x_0) + v(x_0). \tag{12}$$

We will show only that the first limit exists and equals $-\psi(x_0)+v(x_0)$. The second limit is calculated similarly. In the proof, we will make use of the following lemma, sometimes referred to as Gauss' lemma.

Lemma 1.

$$\tilde{v}(x_0) = \int_{\partial\Omega} K(x_0, y)dS_y = \begin{cases} -2, & x_0 \in \Omega, \\ -1, & x_0 \in \partial\Omega, \\ 0, & x_0 \in (\bar{\Omega})^c. \end{cases}$$

Proof. First consider the case where $x_0 \notin \bar{\Omega}$. Define

$$\Phi(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|},$$

the fundamental solution to the Laplace equation in three dimensions. Note that

$$\nabla_y \Phi(x,y) \cdot n_y = \frac{1}{4\pi}\frac{(x-y)\cdot n_y}{|x-y|^3} = \frac{1}{2}K(x,y).$$

Then $\Phi(x_0, y)$ is harmonic in y on a neighborhood of Ω, and by partial integration,

$$\begin{aligned} \int_{\partial\Omega} K(x_0,y)dS_y &= 2\int_{\partial\Omega} \nabla_y \Phi(x_0,y)\cdot n_y \\ &= 2\int_{\Omega} \Delta_y \Phi(x_0 - y)dy = 0. \end{aligned}$$

In case $x_0 \in \Omega$, given some radius $r > 0$ small enough, $\Phi(x_0, y)$ is harmonic in y for $y \in \Omega \backslash B_r(x_0)$. Applying partial integration again, we find

$$\begin{aligned} \int_{\partial\Omega} K(x_0,y)dS_y &= \frac{1}{2\pi}\int_{\partial B_r(x_0)} \frac{(x_0-y)\cdot n}{|x_0-y|^3}dS_y + 2\int_{\Omega\backslash B_r(x_0)} \Delta_y\Phi(x_0,y)dy \\ &= \frac{1}{2\pi}\int_{\partial B_r(x_0)} \frac{(x_0-y)\cdot n}{|x_0-y|^3}dS_y, \end{aligned}$$

with n denoting the outward unit normal to $\partial B_r(x_0)$, i.e.

$$n = \frac{y - x_0}{|y-x_0|}.$$

We can thus calculate the integral and we find

$$\int_{\partial\Omega} K(x_0,y)dS_y = \cancel{2}. = -\frac{1}{2\pi}\int_{|x_0-y|=r} \frac{1}{|x_0-y|^2}dS_y$$

$$= -\frac{1}{2\pi r^2}\, 4\pi r^2 = -2.$$

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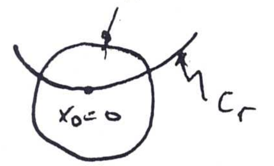
Finally, we treat the case $x_0 \in \partial\Omega$. We may without loss of generality assume $x_0 = (0,0,0)$ and $n_{x_0} = (0,0,-1)$. Again, consider for r small enough $\Omega \backslash B_r(x_0)$, on which $\Phi(x_0, y)$ is harmonic in y. Define

$$C_r := \partial\Omega \backslash \cancel{\partial} B_r(x_0),$$
$$D_r := \partial B_r(x_0) \cap \Omega.$$



Then

$$\int_{C_r} K(x_0,y)dS_y = \frac{1}{2\pi}\int_{D_r} \frac{(x_0-y)\cdot n}{|x_0-y|^3} dS_y + 2\int_{\Omega\backslash B_r(x_0)} \Delta_y \Phi(x_0,y)dy,$$
$$= \frac{1}{2\pi}\int_{D_r} \frac{(x_0-y)\cdot n}{|x_0-y|^3} dS_y,$$

with again

$$n = \frac{y-x_0}{|y-x_0|}.$$

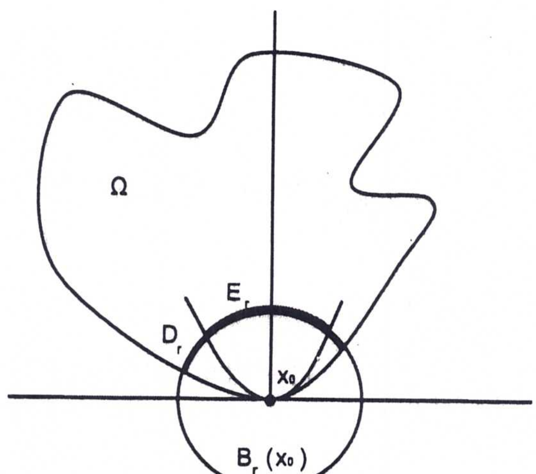
Now,

$$\lim_{r\to 0}\int_{C_r} K(x_0,y)dS_y = \int_{\partial\Omega} K(x_0,y)dS_y,$$

since the integrand on the right-hand side is integrable. On the other hand, defining (see also Fig. 3)

$$E_r = \{z \in \partial B_r(x_0) \,|\, z > C(z_1^2 + z_2^2)\}$$

we have $E_r \subset D_r \subset \partial B_r^+(x_0)$, with $B_r^+(x_0) = B_r(x_0) \cap \{z \in \mathbb{R}^3 | z_3 > 0\}$.



We have

$$\frac{1}{2\pi}\int_{\partial B_r^+(x_0)} \frac{(x_0-y)\cdot n}{|x_0-y|^3} dS_y$$
$$= \frac{1}{2\pi}\int_{E_r} \frac{(x_0-y)\cdot n}{|x_0-y|^3} dS_y$$
$$+ \frac{1}{2\pi}\int_{\partial B_r^+(x_0)\backslash E_r} \frac{(x_0-y)\cdot n}{|x_0-y|^3} dS_y$$
$$= \mathrm{I} + \mathrm{II}.$$

Now

$$\mathrm{II} = -\frac{1}{2\pi}\int_{B_r^+(x_0)\backslash E_r} \frac{dS_y}{|x_0-y|^2}$$
$$= -\frac{1}{2\pi}\int_0^{\theta_0} \frac{2\pi r^2 \cos\theta \, d\theta}{r^2}$$
$$= -\frac{r^2 \sin\theta_0}{r^2}$$

Figure 3: Illustration of the surfaces E_r and D_r, that are used in the last part of the proof of Lemma 1.

Since by monotone convergence

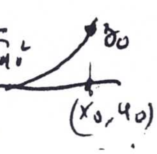
$$\cancel{\lim_{r\to 0}\frac{1}{2\pi}\int_{E_r}\frac{(x_0-y)\cdot n}{|x_0-y|^3}dS_y} = \cancel{\lim_{r\to 0}\frac{1}{2\pi}\int_{B_r^+(x_0)}\frac{(x_0-y)\cdot n}{|x_0-y|^3}}dS_y = -1,$$

Hence $\frac{1}{2\pi}\int_{E_r} \frac{(x_0-y)\cdot n}{|x_0-y|^3} dS_y$

$= \frac{1}{2\pi}\int_{\partial B_r^+(x_0)} \frac{(x_0-y)\cdot n}{|x_0-y|^3} dS_y + o(1) = -1 + o(1)$

$*$ where $\sin\theta_0 = \frac{z_0}{\sqrt{x_0^2+y_0^2}}$

$= C\,\frac{x_0^2+y_0^2}{} \to 0$



Clearly we also find

$$\lim_{r\to 0}\frac{1}{2\pi}\int_{D_r}\frac{(x_0-y)\cdot n}{|x_0-y|^3}dS_y=-1.$$

Putting everything together, we find

$$\int_{\partial\Omega} K(x,y)\,dS_y = \lim_{r\to 0}\int_{C_r} K(x,y)dS_y=-1.$$

□

We can use the lemma to write, for $x\in\Omega$,

$$\begin{aligned}
v(x) &= \int_{\partial\Omega} K(x,y)\psi(y)dy\\
&= \int_{\partial\Omega} K(x,y)(\psi(y)-\psi(x_0))dS_y + \int_{\partial\Omega} K(x,y)\psi(x_0)dS_y\\
&= \int_{\partial\Omega} K(x,y)(\psi(y)-\psi(x_0))dS_y - 2\psi(x_0).
\end{aligned}$$

For $x_0\in\partial\Omega$,

$$\begin{aligned}
v(x_0) &= \int_{\partial\Omega} K(x_0,y)\psi(y)dy\\
&= \int_{\partial\Omega} K(x_0,y)(\psi(y)-\psi(x_0))dS_y + \int_{\partial\Omega} K(x_0,y)\psi(x_0)dS_y\\
&= \int_{\partial\Omega} K(x_0,y)(\psi(y)-\psi(x_0))dS_y - \psi(x_0).
\end{aligned}$$

Consequently,

$$\begin{aligned}
v(x)-v(x_0)+\psi(x_0) &= \int_{\partial\Omega}[K(x,y)-K(x_0,y)](\psi(y)-\psi(x_0))dS_y\\
&=: I_\gamma(x)+J_\gamma(x),
\end{aligned}$$

where $\gamma>0$ and

$$\begin{aligned}
I_\gamma(x) &= \int_{\partial\Omega\cap B_\gamma(x_0)}[K(x,y)-K(x_0,y)](\psi(y)-\psi(x_0))dS_y,\\
J_\gamma(x) &= \int_{\partial\Omega\cap B^c_\gamma(x_0)}[K(x,y)-K(x_0,y)](\psi(y)-\psi(x_0))dS_y.
\end{aligned}$$

Now

$$|I_\gamma(x)|\le\omega(\gamma)\int_{\partial\Omega\cap B_\gamma(x_0)}|K(x,y)-K(x_0,y)|dS_y\le C_3\omega(\gamma),$$

with $C_3>0$ some constant and ω the modulus of continuity of ψ. Since $K(x,y)$ is smooth in y for $y\notin B_\gamma(x_0)$, and ψ is bounded,

$$|J_\gamma(x)|\le C(\gamma)\|\psi\|_\infty|x-x_0|,$$

with $C(\gamma)$ a constant depending on γ. We thus find,

$$|v(x) - v(x_0) + \psi(x_0)| \leq C(\gamma)\|\psi\|_\infty |x - x_0| + C_3\omega(\gamma).$$

This shows the result (11), x_1 since we can first choose γ to make the second term arbitrarily small, and then choose $x \in \Omega$ close enough to x_0 to make the first term arbitrarily small.

Note that the above calculations show that if $v(x)$ defined by (9) for $x \in \mathbb{R}^3 \setminus \partial\Omega$, then $v(x)$ is continuous in $\bar{\Omega}$ and (similarly) also in $\overline{\Omega^c}$.

4 Step 4

We define the operator $T : C(\partial\Omega) \to C(\partial\Omega)$ by

$$(T\psi)(x) = \int_{\partial\Omega} K(x,y)\psi(y)dS_y, \qquad x \in \partial\Omega.$$

For x_1 and x_2 on $\partial\Omega$, $|x_1 - x_2| < \gamma < \delta$, for some γ and with δ as in step 0,

$$\begin{aligned}
|(T\psi)(x_1) - (T\psi)(x_2)| &\leq \int_{\partial\Omega} |K(x_1,y) - K(x_2,y)||\psi(y)|dS_y \\
&\leq \int_{\partial\Omega\cap B_\gamma(x_1)} |K(x_1,y) - K(x_2,y)||\psi(y)|dS_y \\
&\quad + \int_{\partial\Omega\cap B^c_\gamma(x_1)} |K(x_1,y) - K(x_2,y)||\psi(y)|dS_y \\
&=: 2M(2\gamma)\|\psi\|_\infty + C(\gamma)|x_1 - x_2|\|\psi\|_\infty,
\end{aligned}$$

with M a constant defined in (10), and $C(\gamma)$ a constant depending on γ. Therefore, the image of a bounded set under T is equicontinuous (and in particular, T maps to $C(\partial\Omega)$). Moreover, T is a bounded operator, since

$$|T(\psi)(x)| \leq \int_{\partial\Omega} |K(x,y)||\psi(y)|dS_y \leq C_4\|\psi\|_\infty,$$

for some constant $C_4 > 0$. The above inequality also shows that the image of a bounded set under T is equibounded. By the Arzela-Ascoli theorem it follows that T is a compact operator.

5 Step 5

In order to show that the integral equation (3) always has a solution, it suffices to show that $\mathcal{N}(-I + T) = \{0\}$. Let, therefore, $\phi \in C(\partial\Omega)$ be such that $(-I + T)\phi = 0$, and let u be such that

$$u(x) = \int_{\partial\Omega} K(x,y)\phi(y)dy.$$

Then, by the previous steps, u is harmonic inside Ω, and

$$\lim_{\Omega\ni x\to x_0\in\partial\Omega} u(x) = -\phi(x_0) + u(x_0) = -\phi(x_0) + (T\phi)(x_0) = 0.$$

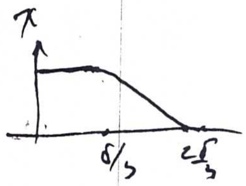
By the maximum principle, $u \equiv 0$ on Ω.

A crucial step in our argument is proving the following claim.

Claim 1. *(The normal derivative is continuous through the boundary)* ~~It holds that~~ We have

$$\left|\frac{\partial u(x_0+tn_{x_0})}{\partial t}\Big|_{t=s} - \frac{\partial u(x_0+tn_{x_0})}{\partial t}\Big|_{t=-s}\right| \to 0,$$

as ~~$s\to 0$~~ $s\downarrow 0$*, uniformly in* x_0.



Proof. As usual, we may assume without loss of generality that $x_0 = (0,0,0)$, and that $n_{x_0} = (0,0,-1)$. We take a function $\chi : [0,\infty) \to [0,1]$, smooth, decreasing, such that $\chi(r) = 1$ for $r < \delta/3$, $\chi = 0$ for $r > 2\delta/3$, and $|\chi'| < 6/\delta$, with δ from step 0. We write

$$\begin{aligned}\phi(y) &= (\phi(y)-\phi(x_0)) + \phi(x_0)\\ &= (\phi(y)-\phi(x_0))\chi(|x_0-y|) + (\phi(y)-\phi(x_0))(1-\chi(|x_0-y|)) + \phi(x_0)\\ &=: \phi_1(y) + \phi_2(y) + \phi(x_0),\end{aligned}$$

with $\phi_1(y) = (\phi(y)-\phi(x_0))\chi(|x_0-y|)$ and $\phi_2(y) = (\phi(y)-\phi(x_0))(1-\chi(|x_0-y|))$. Both are uniformly continuous, and ϕ_1 is supported on $B_{2\delta/3}(x_0)\cap\partial\Omega$, while ϕ_2 vanishes on $B_{\delta/3}(x_0)\cap\partial\Omega$. Now

small

$$\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x_0+tn_{x_0}-y)\cdot n_y}{|x_0+tn_{x_0}-y|^3}\phi_2(y)dS_y$$

is smooth in t, for all t, while

$$\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x_0+tn_{x_0}-y)\cdot n_y}{|x_0+tn_{x_0}-y|^3}\phi(x_0)dS_y$$

small — by Lemma 1

is constant in t for $t>0$ and $t<0$. It suffices therefore to prove the claim for $\tilde u$ instead of u, where

$$\tilde u(x) = \int_{\partial\Omega} K(x,y)\phi_1(y)dy.$$

Close to x_0, we write the surface as a graph of a function $g : B_\delta \to \mathbb{R}$, with the properties as in step 0. The volume element dS_y becomes

$$dS_y = \sqrt{1+g_{y_1}^2+g_{y_2}^2}\,dy_1dy_2,$$

and the normal at y is

$$n_y = \frac{(g_{y_1}, g_{y_2}, -1)}{\sqrt{1+g_{y_1}^2+g_{y_2}^2}}.$$

In that case,

$$\begin{aligned}\tilde u(x_0+tn_{x_0}) &= \frac{1}{2\pi}\int_{\partial\Omega}\frac{(x_0+tn_{x_0}-y)\cdot n_y}{|x_0+tn_{x_0}-y|^3}\phi_1(y)dS_y\\ &= \frac{1}{2\pi}\int_{\partial\Omega}\frac{-y_1g_{y_1}-y_2g_{y_2}+(g+t)}{(y_1^2+y_2^2+(t+g)^2)^{3/2}}\phi_1(y_1,y_2,g(y_1,y_2))dy_1dy_2.\end{aligned}$$

We can calculate the derivative with respect to t by differentiating under the integral (as we have seen in step 2),

$$\begin{aligned}\frac{\partial\tilde u(x_0+tn_{x_0})}{\partial t} &= \int_{B_\delta} dy_1dy_2\tilde\phi(y_1,y_2)\Big[\frac{y_1^2+y_2^2+(t+g)^2}{(y_1^2+y_2^2+(t+g)^2)^{5/2}}\\ &\qquad - \frac{3(-y_1g_{y_1}-y_2g_{y_2}+(t+g))(t+g)}{(y_1^2+y_2^2+(t+g)^2)^{5/2}}\Big]\\ &= I_1(t)+I_2(t),\end{aligned}$$

where

$$\tilde{\phi}(y_1, y_2) = \phi_1(y_1, y_2, g(y_1, y_2)),$$

and

$$I_1(t) = \int_{B_\delta} \frac{dy_1 dy_2 \tilde{\phi}(y_1, y_2)}{(y_1^2 + y_2^2 + (t+g)^2)^{5/2}} \left[y_1^2 + y_2^2 - 2t^2 - 4tg + 3t(y_1 g_{y_1} + y_2 g_{y_2}) \right]$$

$$I_2(t) = \int_{B_\delta} \frac{dy_1 dy_2 \tilde{\phi}(y_1, y_2)}{(y_1^2 + y_2^2 + (t+g)^2)^{5/2}} \left[-2g^2 + 3g(y_1 g_{y_1} + y_2 g_{y_2}) \right]$$

From step 0 it immediately follows that

$$| -2g^2 + 3g(y_1 g_{y_1} + y_2 g_{y_2})| \leq 8 \max(1, C^2)(y_1^2 + y_2^2)^2.$$

Consequently,

$$|I_2(t) - I_2(0)| \to 0, \qquad t \to 0,$$

where the convergence is uniform in x_0. It then also follows that

$$|I_2(t) - I_2(-t)| \to 0, \qquad t \to 0,$$

uniformly in x_0.

We will now consider the term $I_1(t)$. We substitute $tw_i = y_i$. Then

$$I_1(t) = \frac{t^2 t^2}{t^5} \int_{B_{\delta/t}} dw_1 dw_2 \tilde{\phi}(tw_1, tw_2) \frac{w_1^2 + w_2^2 - 2 - 4g/t + 3(w_1 g_{y_1} + w_2 g_{y_2})}{(w_1^2 + w_2^2 + (1 + g/t)^2)^{5/2}}.$$

We factor out $(w_1^2 + w_2^2 + 1 + g^2/t^2)$ in the denominator

$$\begin{aligned} I_1(t) &= \frac{1}{t} \int_{B_{\delta/t}} dw_1 dw_2 \frac{\tilde{\phi}(tw_1, tw_2)(w_1^2 + w_2^2 - 2 - 4g/t + 3(w_1 g_{y_1} + w_2 g_{y_2})}{(w_1^2 + w_2^2 + 1 + g^2/t^2)^{5/2}(1 + \frac{2g/t}{w_1^2 + w_2^2 + 1 + g^2/t^2})^{5/2}} \\ &= \frac{1}{t} \int_{B_{\delta/t}} dw_1 dw_2 \frac{\tilde{\phi}(tw_1, tw_2)}{(w_1^2 + w_2^2 + 1 + g^2/t^2)^{5/2}(1 + \frac{2g/t}{w_1^2 + w_2^2 + 1 + g^2/t^2})^{5/2}} \times \\ &\qquad \left[-2 + w_1^2 + w_2^2 + F(t, w_1, w_2) \right], \end{aligned}$$

where we have defined

$$F(t, w_1, w_2) := -4 \frac{g(tw_1, tw_2)}{t} + 3\left(w_1 g_{y_1}(tw_1, tw_2) + w_2 g_{y_2}(tw_1, tw_2)\right).$$

By step 0 it follows that

$$|F(t, w_1, w_2)| \leq 10C|t|(w_1^2 + w_2^2)$$

Also define $G(t, w_1, w_2)$ such that

$$\left(1 + \frac{2g/t}{w_1^2 + w_2^2 + 1 + g^2/t^2}\right)^{-5/2} = 1 + G(t, w_1, w_2).$$

Then, again by step 0, for t small enough,

$$|G(t, w_1, w_2)| \leq 10C|t|.$$

Thus,

$$I_1(s) = \frac{1}{s}\int_{B_{\delta/s}} dw_1 dw_2 \frac{\tilde{\phi}(sw_1, sw_2)[-2 + w_1^2 + w_2^2 + F(s, w_1, w_2)](1 + G(s, w_1, w_2))}{(w_1^2 + w_1^2 + 1 + g^2/s^2)^{5/2}}$$

For $t = -s$,

$$I_1(-s) = \frac{1}{s}\int_{B_{\delta/s}} dw_1 dw_2 \frac{\tilde{\phi}(sw_1, sw_2)[-2 + w_1^2 + w_2^2 + F(-s, -w_1, -w_2)](1 + G(-s, -w_1, -w_2))}{(w_1^2 + w_1^2 + 1 + g^2/s^2)^{5/2}}$$

We compare $I_1(s)$ and $I_1(-s)$ for $s > 0$,

$$|I(s) - I(-s)| \leq \frac{1}{s}\int_{B_{\delta/s}} dw_1 dw_2 \frac{C s\tilde{\phi}(sw_1, sw_2)(w_1^2 + w_2^2 + 1)}{(w_1^2 + w_2^2 + 1)^{5/2}}.$$

Now define

$$E(s) = \int_{B_{\delta/s} \setminus B_{\delta/\sqrt{s}}} \frac{w_1^2 + w_2^2}{(w_1^2 + w_2^2 + 1)^{5/2}} dw_1 dw_2,$$

then $E(s) \to 0$ as $s \to 0$. We have the bound

$$|I_1(s) - I_1(-s)| \leq C_5(\omega(\sqrt{s}) + \|\tilde{\phi}\|_\infty E(s)),$$

with ω the modulus of continuity of ϕ_1, and $C_5 > 0$ a constant that does not depend on x_0.

□

Recall from step 0 the definitions of the domain Ω_t and the smooth hypersurface S_t,

$$\partial\Omega_t = S_t = h(\partial\Omega, t).$$

For some radius R big enough and $s < \delta$,

$$\begin{aligned}\int_{B_R \setminus \Omega_s} |\nabla u|^2 &= \int_{\partial B_R} u\frac{\partial u}{\partial n} + \int_{\partial\Omega_s} u\frac{\partial u}{\partial n} + \int_{B_R \setminus \Omega_t} u\Delta u \\ &= \int_{\partial B_R} u\frac{\partial u}{\partial n} - \int_{\partial\Omega} u(h(y, s))\frac{\partial u(h(y,t))}{\partial t}\Big|_{t=s} J_s(y) dS_y,\end{aligned}$$

where $J_s(y)$ is the Jacobian, of the transformation $z = h(y, s)$, that is, for all $\varphi \in C^\infty(\mathbb{R}^3)$,

$$\int_{\partial\Omega_s} \varphi(z) dS_z = \int_{\partial\Omega} \varphi(h(y, s)) J_s(y) dS_y.$$

Now

~~It holds that~~ $J_s(y)$ converges to 1 as $s \to 0$ uniformly in y. By the form of the solution,

$$u(x) = \int_{\partial\Omega} K(x, y)\phi(y) dS_y,$$

it is clear that the integral over ∂B_R vanishes in the limit $R \to 0$. In the limit $t \to 0$, the integral over $\partial\Omega_s$ vanishes, too, since by the claim and the facts that $u \equiv 0$ on Ω, and $J_s(y)$ converges to 1 as $s \to 0$ uniformly in y,

$$\frac{\partial u(x_0 + tn_{x_0})}{\partial t}\Big|_{t=s} \to 0, \text{ as } s \to 0,$$

so that

$$\frac{\partial u(x_0 + tn_{x_0})}{\partial t} \equiv 0$$

for all $t < 0$ and small

and as $u(x) \to 0$ as $|x| \to \infty$ we must have $u(x) \equiv 0$ in Ω^c

uniformly in x_0. It follows that

$$\int_{\mathbb{R}^3 \setminus \Omega} |\nabla u|^2 = 0.$$

Hence, u is constant ~~on~~ in $\bar{\Omega}^c$. By (5), ~~ϕ is a constant. But then $u \equiv 0$ on $\bar{\Omega}^c$~~, so ϕ is in fact equal to 0. it then follows that $\phi(x) \equiv 0$ on $\partial\Omega$, and we are done.

Question The method used above is called the "double layer" potential method

Could you solve the Dirichlet problem in this fashion, but with a single layer potential? For what problem would a single layer potential be suited?

6 References

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