1. Consider the uniform slow motion with speed U of a viscous fluid past a spherical bubble of radius a, filled with air. Do this by modifying the Stokes flow analysis for a rigid sphere as follows. The no slip condition is to be replaced on $r = a$ by the condition that both u_r and the tangential stress $\sigma_{r\theta}$ vanish. (This latter condition applies since there is no fluid within the bubble to support this stress.) Show in particular that

$$\Psi = \frac{U}{2}(r^2 - ar)\sin^2\theta$$

and that the drag on the bubble is $D = 4\pi\mu U a$. Note: On page 235 of Batchelor see the analysis for a bubble filled with a second liquid of viscosity $\bar{\mu}$. The present problem is for $\bar{\mu} = 0$.

2. Consider two-dimensional Stokes flow past a circular cylinder of radius a. Show that the problem reduces to the biharmonic equation for the two-dimensional stream function ψ,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0,$$

with conditions $\partial\psi/\partial r = \partial\psi/\partial\theta = 0$ on $r = a$ and $\psi \sim Ur\sin\theta$ as $r \to \infty$. Seeking a solution of the form $f(r)\sin\theta$, show that this leads to

$$f = Ar^3 + Br\log r + Cr + D/r$$

and hence that there is no solution of the required form. This is *Stokes' Paradox*, as discussed in class.

3. Construct a set of solutions of the Stokes equations in three dimensions, having the form

$$u_i = \varepsilon_{ijk}\frac{\partial\chi}{\partial x_j}\Omega_k$$

where $\varepsilon_{ijk} = \pm 1$ for ijk and even (odd) permulation of 123 and is otherwise $= 0$, and Ω_k is a constant vector. What is the corresponding pressure field, if we assume that p must vanish at infinity? Use the form to find the flow field generated by a rigid sphere of radius a spinning with angular velocity Ω_k in a very viscous fluid, such that $\mathbf{u} = 0$ at infinity. Show that the torque exerted by sphere on fluid is $8\pi\Omega a^3\mu$. (Solutions of the above form, together this those given in class, form a complete set of Stokes flows in three dimensions.)

4. Prove that Stokes flow past a given, rigid body is unique, as follows. Show if $p_1, \mathbf{u}_1$ and $p_2, \mathbf{u}_2$ are two solutions of

$$\nabla p - \mu\nabla^2\mathbf{u} = 0, \nabla\cdot\mathbf{u} = 0,$$

satisfying $u_i = -U_i$ on the body and

$$\mathbf{u} \sim= O(1/r), \frac{\partial u_i}{\partial x_j}, p \sim O(1/r^2)$$

as $r \to \infty$, then the two solutions must agree. (Hint: Consider the integral of $\partial/\partial x_i(w_j\partial w_j/\partial x_i)$ over the region exterior to the body, where $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$.)

5. Two small spheres of radius a and density ρ_s are falling in a viscous fluid with centers at P and Q. The line PQ has length $L \gg a$ and is perpendicular to gravity. Using the Stokeslet approximation to the Stokes solution past a sphere, and assuming that each sphere sees the unperturbed Stokes flow of the other sphere, show that the spheres fall with the same speed

$$U \approx U_s(1 + ka/L + O(a^2/L^2)),$$

and determine the number k. Here $U_s = 2a^2g/9\nu(\rho_s/\rho - 1)$ is the settling speed of a single sphere in Stokes flow.