1. Consider a Navier-Stokes fluid of constant ρ, μ, no body forces. Consider a motion in a fixed bouned domain V with no-slip condition on its rigid boundary. Show that

$$dE/dt = -\Phi, E = \int_V \rho|\mathbf{u}|^2/2dV, \Phi = \mu \int_V (\nabla \times \mathbf{u})^2 dV.$$

This shows that for such a fluid kinetic energy is converted into heat at a rate $\Phi(t)$. This last function of time gives the net *viscous dissipation* for the fluid contained in V. (Hint: $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$.)

2. In two dimensions, with streamfunction ψ, where $(u, v) = (\psi_y, -\psi_x)$, show that the incompressible Navier-Stokes equations without body forces for a fluid of constant ρ, μ reduce to

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{(\partial(\psi, \nabla^2\psi)}{\partial(x, y)} - \nu\nabla^4\psi = 0.$$

In terms of ψ, what are the boundary conditions on a rigid boundary if the no-slip condition is satisfied there?

3. Consider the steady 2D flow of a layer of viscous incompressible fluid under gravity down an inclined plane. You may assume the streamlines are parallel to the plane, and that $(u, v) = (u(y), 0)$, where the x-axis is parallel to the plane (see the figure). Write down the equations for the flow, assuming constant ρ, μ. Solve for the pressure and for u, requiring that$p = p_0$=constant and $\mu du/dy = 0$ at the free surface adjacent to the air. (The latter condition imposes zero stress at the free surface). Compute the volume flux of fluid down the plane as a function of $\nu = \mu/\rho$, gravity g, and the layer thickness H.

4. Find the time-periodic 2D flow in a channel $-H < y < H$, filled with viscous incompressible fluid, given that the pressure gradient is $dp/dx = A + B\cos(\omega t)$, where A, B, ω are constants. This is an oscillating 2D Poiseuille flow. You may assume that $u(y, t)$ is even in y and periodic in t with period $2\pi/\omega$.

5. (See Batchelor p. 285) Consider the steady 2D Navier-Stokes equations, ρ, μ constant. We seek an exact solution describing viscous flow into a stagnation point (see the figure). Recall that the inviscid stagnation-point flow has the velocity field $(u, v) = (x, -y)$. We look for a Navier-Stokes flow with $(u, v) = (xf'(y), -f(y))$, with $f'(\infty) = 1$. (Assume also that f'' and f''' vanish as $y \to \infty$.) What conditions should be satisfied by f at $y = 0$ to impose the no-slip condition there? Deduce the form of p and find the equations satisfied by $f(y)$. The solution provides an interesting example of a boundary layer of constant thickness. What is the rough scale of the thickness as a function of $\nu = \mu/\rho$?