1. This problem will study flow past a slender axisymmetric body whose surface is given (in cylindrical polar coordinates), by $r = R(z), 0 \leq z \leq L$. Here $R(z)$ is continuous, and positive except at $0, L$ where it vanishes. By "slender" we mean that $\max_{0 \leq x \leq L} R << L$. The body is placed in the uniform flow $(u_z, u_r, u_\theta) = (U, 0, 0)$. We are interested in the steady, axisymmetric potential flow past the body. It can be shown that such a body perturbs the free stream by only a small amount, so that in particular, $u_z \approx U$ everywhere. On the other hand the flow must be tangent at the body, which implies $\phi_r(z, R(z)) \approx U dR/dz, 0 < z < L$.

We look for a representation of ϕ as a distribution of sources with strength $f(z)$. Thus

$$\phi(z, r) = -\frac{1}{4\pi} \int_0^L \frac{f(\zeta)}{\sqrt{(z-\zeta)^2 + r^2}} d\zeta.$$

(a) Compute $\frac{\partial \phi}{\partial r}$, and investigate the resulting integral as $r \to 0, 0 < z < L$. Argue that the dominant contribution comes near $\zeta = z$, and hence show that $\frac{\partial \phi}{\partial r} \approx \frac{1}{4\pi} \frac{f(z)}{r} \int_{-\infty}^{+\infty} (1 + s^2)^{-3/2} ds$ for $r << L$.

(b) From the above tangency condition, deduce that $f(z) \approx dA/dz$ where $A(z) = \pi R^2$ is the cross-sectional area of the body.

(c) By expanding the above expression for ϕ for large z, r, show that in the neighborhood of infinity

$$\phi \approx \frac{1}{4\pi} \frac{z}{(z^2 + r^2)^{3/2}} \int_0^L A(\zeta) d\zeta, z^2 + r^2 \to \infty.$$

2. Compute, using the Blasius formula, the force exerted by a simple source at a point $c = be^{i\theta}$ on a circular cylinder of radius $a < b$. Recall that the complex potential for a simple source at c is $w = (Q/2\pi) \log(z - c)$.(Hint: Use the circle theorem, then compute the force from the residue at c.) Verify that the cylinder is pulled toward the source.

3. Using the method of Blasius for obtaining moment, as outlined in class, show that the moment of a cylinder in 2D potential flow is given by

$$M = -\frac{1}{2} \rho \Re[\int_C z(dw/dz)^2 dz]$$

where $\Re$ denotes the real part and C is any simple contour about the body. Using this, verify that the circular cylinder flow with vortex of strength Γ at its center experiences zero moment. (Use the residue method.).

4. Investigate the pressure distribution on a lifting flat plate with circulation Γ. Give an expression for the pressure as a function of position, for both top and bottom surfaces, with arbitrary Γ. Describe the singularities in pressure at the two end-points of the plate, $(x, y) = (\pm 2l, 0)$. Apply the Kutta condition, and show that in this case the rear singularity disappears and

$$p(x, 0) = p(2l, 0) - \frac{1}{2} \rho U^2 \big[\big(\frac{2l - x}{2l + x} \big) \sin^2(\alpha) \pm 2 \big(\frac{2l - x}{2l + x} \big)^{1/2} \sin\alpha \cos\alpha \big],$$

where the upper/lower sign refers to the upper/lower side of the plate. (Note that l in this problem is the same as the parameter a in the lecture notes, i.e. the cylinder radius.)