Fluid Dynamics I PROBLEM SET 3 Due October 6, 2003

1. Consider a fluid of constant density in two dimensions with gravity, and suppose that the vorticity $u_y - v_x$ is everywhere constant and equal to ω. Show that the velocity field has the form $(u, v) = (\phi_x + \chi_y, \phi_y - \chi_x)$ where ϕ is harmonic and χ is any function of x, y (independent of t), satisfying $\nabla^2\chi = -\omega$. Show further that

$$\nabla(\phi_t + \frac{1}{2}q^2 + \omega\psi + p/\rho + gz) = 0$$

where where ψ is the streamfunction for $\vec{u}$, i.e. $\vec{u} = (\psi_y, -\psi_x)$, and $q^2 = u^2 + v^2$.

2. Show that, for an incompressible fluid, but one where the density can vary independently of pressure (e.g. salty seawater), the vorticity equation is

$$\frac{D\omega}{Dt} = \omega \cdot \nabla\vec{u} + \rho^{-2}\nabla\rho \times \nabla p.$$

Interpret the last term on the right physically. (e.g. what happens if lines of constant p are $y =$ constant and lines of constant ρ are $x - y =$constant?). Try to understand how the term acts as a source of vorticity, i.e. causes vorticity to be created in the flow.

3. For steady two-dimensional flow of a fluid of constant density, we have

$$\rho\mathbf{u} \cdot \nabla\vec{u} + \nabla p = 0, \nabla \cdot \vec{u} = 0.$$

Show that, if $\vec{u} = (\psi_y, -\psi_x)$, these equations imply

$$\nabla\psi \times \nabla(\nabla^2\psi) = 0.$$

Thus, show that a solution is obtained by giving a function $H(\psi)$ and then solving $\nabla^2\psi = H'(\psi)$. Show also that the pressure is given by $\frac{p}{\rho} = H(\psi) - \frac{1}{2}(\nabla\psi)^2$+constant.

4. Prove *Ertel's theorem* for a fluid of constant density: If f is a scalar material invariant, i.e. $Df/Dt = 0$, then $\omega \cdot \nabla f$ is also a material invariant, where $\omega = \nabla \times \vec{u}$ is the vorticity field.

5. A steady *Beltrami flow* is a velocity field $\vec{u}(\vec{x})$ for which the vorticity is always parallel to the velocity, i.e. $\nabla \times \vec{u} = f(\vec{x})\vec{u}$ for some scalar function f. Show that if a steady Beltrami field is also the velocity field of an inviscid fluid of constant density, the necessarily f is constant on streamlines. What is the corresponding pressure? Show that $\vec{u} = (Bsiny + C\cos z, C\sin z + A\cos x, A\sin x + B\cos y)$ is such a Beltrami field with $f = -1$. (This last flow an example of a velocity field yielding chaotic particle paths. This is typical of 3D Beltrami flows with constant f, according to a theorem of V. Arnold.)