1. Find Lagrangian coordinates for the following Eulerian velocity fields:

$$(a)\ \ u = x, v = -y; \quad (b)\ \ u = y + t^2, v = -x$$

In each case express your answer as functions $x(a, b, t, t_0), y(a, b, t, t_0)$ where $x(a, b, t_0, t_0) = a, y(a, b, t_0, t_0) = b$. For each flow describe the instantaneous streamline through $a = b = 1$ at t=1, the particle path for the particle which is at the $a = b = 1$ when $t = t_0 = 1$, and the streak line from $a = b = 1$ for $t_0 < t = 0$.

2. Consider the "point vortex " flow in two dimensions,

$$(u, v) = UL(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}),\ x^2 + y^2 \neq 0,$$

where U, L are reference values of speed and length. (a) Show that the Lagrangian coordinates for this flow may be written

$$x(a, b, t) = R_0 \cos{(\omega t + \theta_0)},\ y(a, b, t) = R_0 \sin{(\omega t + \theta_0)}$$

where $R_0^2 = a^2 + b^2, \theta_0 = \arctan{(b/a)}$, and $\omega = UL/R_0^2$. (Note that on a particle path $xdx + ydy = 0$.)(b) Consider, at $t = 0$ a small rectangle of marked fluid particles determined by the points $A(L, 0), B(L + \Delta x, 0), C(L + \Delta x, \Delta y), D(L, \Delta y)$. If the points move with the fluid, once point A returns to its initial position what is the shape of the marked region? Since $(\Delta x, \Delta y)$ are small, you may assume the region remains a parallelogram. Do this, first, by computing the entry $\partial y/\partial a$ in the jacobian, evaluated at $A(L, 0)$. Then verify your result by considering the "lag" of particle B as it moves on a slightly larger circle at a slightly slower speed, relative to particle A, for a time taken by A to complete one revolution.

3. As was noted in class, Lagrangian coordinates can use any unique labeling of fluid particles. To illustrate this, consider the Lagrangian coordinates in two dimensions

$$x(a, b, t) = a + \frac{1}{k} e^{kb} \sin k(a + ct),\ y = b - \frac{1}{k} e^{kb} \cos k(a + ct),$$

where k, c are constants. Note here a, b are *not* equal to x, y for any t_0. By examining the determinant of the Jacobian, verify that we have a unique labeling of fluid particles provided that $b \neq 0$. What is the situation if $b = 0$? (These waves, which were discovered by Gerstner in 1802, represent surface gravity waves moving with speed c, $c^2 = g/k$, where g is the acceleration of gravity. They do not have any simple Eulerian representation.)

4. In one dimension, the Eulerian velocity is given to be $u(x,t) = 2x/(1+t)$. (a) Find the Lagrangian coordinate $x(a,t)$. (b) Find the Lagrangian velocity as a function of a,t. (c) Find the jacobian $\partial x/\partial a = J$ as a function of a,t. (d) If density satisfies $\rho(a,0) = a$ and mass is conserved, find $\rho(a,t)$ using the Lagrangian form of mass conservation. (e) From (a) and (d) evaluate ρ as a function of x,t, and verify that the Eulerian conservation of mass equation is satisfied by $\rho(x,t), u(x,t)$.

5. For the stagnation-point flow $\mathbf{q} = (u,v) = U/L(x,-y)$, show that a fluid particle in the first quadrant which crosses the line $y = L$ at time $t = 0$, crosses the line $x = L$ at time $t = \frac{L}{U}\log(UL/\psi)$ on the streamline $Uxy/L = \psi$. Do this two ways. First, consider a line integral of $\mathbf{q}\cdot\mathbf{ds}/(u^2+v^2)$ along a streamline. Second, use Lagrangian variables.

6. Let V_t denote a material fluid volume in three-dimensions. Prove that, for any smooth function $\vec{g}(\vec{x},t)$,

$$\frac{d}{dt}\int \rho\vec{g}\, dV_t = \int \rho D\vec{g}/Dt\, dV_t.$$

Here ρ is the density, satisfying the mass conservation equation $\rho_t + \nabla\cdot(\rho\vec{u}) = 0$, and $D/Dt = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla$. (Hint: $dV_t = det(\mathbf{J})dV_0$. from the result for $det(\mathbf{J})$ proved in class.)