Fluid Dynamics I PROBLEM SET 10 Due by the last class. This is the last problem set.

The FINAL EXAM will be at the class time on December 15.

1. Viscous fluid is contained between two rigid boundaries, $z = 0$ and $z = H$. The lower plane is at rest, while the upper plane rotates about a vertical axis with constant angular velocity Ω. The Reynolds number $\Omega H^2/\nu$ is small, so Stokes' theory applies. Show that Stokes' equations in cylindrical polar coordinates admit a purely rotary flow solution of the form

$$(u_r, u_\theta, u_z) = (0, u_\theta(r, z), 0), \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)u_\theta = 0.$$

Write down boundary conditions which u_θ must satisfy at $z = 0, H$. Then seek a solution of the form $u_\theta = rf(z)$. Find the θ-component of stress on the upper plane is $-\mu\Omega r/H$ as a function of r. Find the torque exerted by the fluid on the disk $r < a$ of the upper wall.

2. *Oseen's equations* are sometimes proposed as a model of the Navier-Stokes, equations, in the study of steady viscous flow past a body. Oseen's equations, for a flow with velocity $(U, 0, 0)$ at infinity, are

$$U\frac{\partial \mathbf{u}}{\partial x} + \frac{1}{\rho}\nabla p - \nu\nabla^2\mathbf{u} = 0, \nabla \cdot \mathbf{u} = 0.$$

(a) Show that in this model, if $\nu = 0$, the vorticity is a function of y, z alone.

(b) For the Oseen model, and for a flat plate aligned with the flow, carry out Prandtl's simplifications for deriving the boundary-layer equations in two dimensions, given that the boundary condition of no slip is retained at the body. That is find the form of the boundary layer on a flat plate of length L aligned with the flow at infinity, according to Oseen's model, and show that in the boundary layer the the x-component of velocity, u, satisfies

$$U\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial y^2}.$$

What are the boundary conditions on u for the flat-plate problem? Find the solution, by assuming that u is a function of $y\sqrt{\frac{U}{\nu x}}$, for $0 < x < L$.

(c) Compute the drag coefficient of the plate (drag divided by $\rho U^2 L$, and remember there are two sides), in the Oseen model.

3. What are the boundary-layer equations for the boundary-layer on the front portion of a circular cylinder orf radius a, when the free stream velocity is $(-U, 0, 0)$ (see figure below)? (Use cylindrical polar coordinates). What is the role of the pressure in the problem? Be sure to include the effect of the pressure as an explicit function in your momentum equation, the latter being determined by the potential flow past a circular cylinder studied previously. Show that, by defining $x = a\theta, \bar{y} = (r - a)\sqrt{R}$ in the derivation of the boundary-layer equations, the equations are equivalent to a boundary layer on a flat plate aligned with the free stream, in rectangular coordinates, but with pressure a given function of x.

4. For a *cylindrical* jet emerging from a hole in a plane wall, we have a problem analogous to the 2D jet considered in class (see figure 2). Consider only the boundary-layer limit. (a) Show that

$$\frac{\partial}{\partial z}(u_z^2) + \frac{1}{r}\frac{\partial}{\partial r}(ru_r u_z) - \frac{\nu}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}) = 0,$$

and hence that the momentum M is a constant, where

$$M = 2\pi\rho\int_0^\infty r u_z^2 dr.$$

(b) Letting $(u_z, u_r) = (1/r)(\psi_r, -\psi_z)$ where $\psi(0, z) = 0$ show that we must have $\psi = zf(\eta),\ \eta = r^2/z^2$. Determine the equation for f and thus show that the boundary-layer limit has the form

$$f = 4\nu\frac{\eta}{\eta + \eta_0},$$

where η_0 is a constant. Express η_0 in terms of M, the momentum flux of the jet defined above.

5.consider the Prandtl boundary-layer equations with $U(x) = 1/x$, so $p(x)/\rho = p_\infty - 1/(2x^2)$. Verify that the similarity solution has the form $\psi = f(\eta), \eta = y/x$. Find the equation for f. Show that there is no continuously differentiable solution of the equation which satisfies $f(0) = f'(0) = 0$ and $f' \to 1, f'' \to 0$ as $\eta \to \infty$. (Hint: Obtain an equation for $g = f'$.)