A fast, accurate, and easy to implement quadrature scheme for singular integrals in axisymmetric geometries

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Many applications in magnetic confinement fusion require the efficient calculation of surface integrals with singular integrands. The singularity subtraction approaches typically used to handle such singularities are complicated to implement and low order accurate. In contrast, we demonstrate that the Kapur-Rokhlin quadrature scheme is well-suited for the logarithmically singular integrals encountered for toroidally axisymmetric confinement system, is easy to implement and high order accurate. As an illustration, we show how to apply this quadrature scheme for the efficient and accurate calculation of the normal component of the magnetic field due to the plasma on the plasma boundary, via the virtual casing principle.

1. Introduction

Integral formulations and integral equations are effective and popular tools for magnetostatic and magnetohydrodynamic problems in magnetic confinement fusion (Shafranov & Zakharov 1972; Zakharov 1973; Freidberg \textit{et al.} 1976; Merkel 1986; Hirshman \textit{et al.} 1986; Hirshman & Neilson 1986; Chance 1997; Ludwig \textit{et al.} 2006, 2013; Lazerson \textit{et al.} 2013; Drevlak \textit{et al.} 2018; O’Neil & Cerfon 2018; Malhotra \textit{et al.} 2019a; Pustovitov & Chukashev 2021). They have intuitive physical interpretations (Shafranov & Zakharov 1972; Zakharov 1973; Hirshman & Neilson 1986; Lazerson \textit{et al.} 2013; Hanson 2015; Pustovitov & Chukashev 2021), provide geometric flexibility (Merkel 1986; Hirshman \textit{et al.} 1986; Chance 1997; O’Neil & Cerfon 2018; Malhotra \textit{et al.} 2019a), and often reduce the dimension of the unknown quantities one solves for, thus reducing the number of unknowns (Merkel 1986; Hirshman \textit{et al.} 1986; Chance 1997; O’Neil & Cerfon 2018; Malhotra \textit{et al.} 2019a). However, there typically is a price to pay for these advantages. Integral formulations often involve singular integrands, which are subtle to handle numerically (Freidberg \textit{et al.} 1976; Merkel 1986; Atkinson 1997; Chance 1997; Ludwig \textit{et al.} 2006, 2013; Klöckner \textit{et al.} 2013; Kress 2014; Ricketson \textit{et al.} 2016; Malhotra \textit{et al.} 2019a; Landreman & Boozer 2016). The numerical difficulty of integrating these singular integrands depends on the nature of the singularity, the distribution of sources, and the relative location of the evaluation points (often known as target points or observation points) with respect to the sources. In this article, we will focus on the common situation in which we are trying to evaluate layer potentials at the source locations. This is for example the standard situation when applying Green’s identity (Freidberg \textit{et al.} 1976; Merkel 1986; Hirshman \textit{et al.} 1986; Pustovitov 2008; Lee \textit{et al.} 2015; Malhotra \textit{et al.} 2019b).

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In the fusion community, the numerical difficulty due to the singularity of the integrand is usually addressed via the method of singularity subtraction (Freidberg et al. 1976; Merkel 1986; Chance 1997; Ludwig et al. 2006, 2013). The method is robust, but leads to a quadrature scheme with low order convergence. Furthermore, it is complicated to implement, and the chances of making mistakes in the derivation of the quadrature scheme or its numerical implementation are high. The purpose of our work is to demonstrate that for the singular layer potential integrals encountered in axisymmetric confinement devices, which can be reduced to line integrals of singular periodic functions, the Kapur-Rokhlin quadrature scheme (Kapur & Rokhlin 1997) is as simple to implement as the trapezoidal rule, and is a scheme with high order convergence, leading to low error for few quadrature points.

As discussed above, there are many situations in the study of axisymmetric magnetic confinement fusion devices for which the simplicity and accuracy of the Kapur-Rokhlin scheme could be demonstrated. For this article, we choose to focus on the evaluation of single-layer and double-layer potentials, which are the layer potentials appearing in Green’s identity, and which we will define precisely in section 2. Our first numerical test is a numerical verification of an identity for the double-layer potential. Our second numerical test focuses on the single-layer potential, which we evaluate for an application of the virtual casing principle, to calculate the normal component of the magnetic field due to the plasma current at points on the plasma boundary (Shafranov & Zakharov 1972; Zakharov 1973; Hanson 2015).

The structure of this article is as follows. We introduce our mathematical notation and layer potentials in Section 2. In Section 3, we give a brief summary of the virtual casing principle, discuss the mathematical difficulties associated with the numerical evaluation of the virtual casing integral, and describe our method for addressing these difficulties in toroidally axisymmetric geometries. We prove in Section 4 that the integrands we consider in this article are logarithmically singular, and can therefore be integrated to high accuracy with the Kapur-Rokhlin quadrature scheme, which we present in Section 5. We demonstrate the accuracy and high order of convergence of the scheme for an application of the virtual casing principle and for the evaluation of a double-layer potential in Section 6, and summarize our work in Section 7.

2. Mathematical background

2.1. Description of toroidal volumes and surfaces

Throughout our discussion of toroidal geometries, we shall make use of the standard, right-handed cylindrical coordinates \((r, \phi, z)\). At a point with toroidal angle \(\phi\), we write the orthonormal unit vectors as \(e_r(\phi)\), \(e_\phi(\phi)\), and \(e_z\). With this notation, we emphasize the fact that the radial and azimuthal unit vectors depend on the toroidal angle.

In this article, we will focus on axisymmetric geometries, which means that we shall only consider surfaces and volumes of revolution. We take the \(z\)-axis as the axis of revolution and define a simple closed curve \(\gamma\) in the \((r, z)\) plane. By rotating this curve about the \(z\)-axis through the toroidal angle \(\phi \in [0, 2\pi]\), we obtain a closed surface of revolution \(\Gamma\). Its interior \(\Omega\) is the corresponding volume of revolution. We refer to \(\gamma\) as the generating curve of \(\Gamma\). It is parameterized by a single variable \(t\), which we assume has period \(L\). We denote the components of \(\gamma\) in the \((r, z)\) plane by \((r(t), z(t))\), and we identify a point \(y \in \Gamma\) by its toroidal revolution angle \(\phi\) and its generating curve parameter \(t\). Correspondingly, we often write \(y = y(\phi, t)\) to stress this parameterization. Finally, we assume that \(\gamma\) is oriented so that the vector \(n(y(\phi, t)) = (\partial y/\partial \phi) \times (\partial y/\partial t)/J(t)\) is
the unit outward normal to $\Omega$ at $y$. The quantity $J(t) = ||(\partial y/\partial \phi) \times (\partial y/\partial t)||$ is the Jacobian of the parameterization.

### 2.2. Single-layer and double-layer potentials for axisymmetric geometries

Layer potentials are fundamental tools in representing solutions to the partial differential equations that arise in magnetostatic and magnetohydrodynamic calculations for magnetic confinement fusion (Merkel 1986; Chance 1997; Ludwig et al. 2006, 2013; Landreman & Boozer 2016; Drevlak et al. 2018). Given a surface $\Gamma$ and a free-space Green’s function $(x, y) \mapsto G(x, y)$ for a partial differential equation, the single layer operator $S$ and the double layer operator $D$ are defined by (Guenther & Lee 1996)

$$[S\sigma](x) = \int_\Gamma G(x, y)\sigma(y)d\Gamma(y)$$

and

$$[D\sigma](x) = \int_\Gamma \frac{\partial G(x, y)}{\partial n(y)}\sigma(y)d\Gamma(y),$$

respectively. The function $\sigma$ is called the density function in this representation. Functions expressible as single-layer and double-layer potentials automatically satisfy the partial differential equation associated with the Green’s function everywhere except the boundary $\Gamma$.

The case of Laplace’s equation in three dimensions is particularly prevalent in magnetostatic and magnetohydrodynamic settings (Merkel 1986; Chance 1997; Landreman & Boozer 2016; Drevlak et al. 2018). Here, the Green’s function is

$$G(x, y) = \frac{1}{4\pi ||x - y||},$$

and functions expressible as $S\sigma$ or $D\sigma$ are harmonic on $\mathbb{R}^3 \setminus \Gamma$. Assuming the density $\sigma$ is also axisymmetric in the sense that $\partial \sigma/\partial \phi = 0$, one can analytically compute the part of the surface integral over the revolution angle $\phi \in [0, 2\pi]$. The resulting single-layer and double-layer integrals then are one-dimensional line integrals, and the resulting Green’s function in the integrand can be expressed in terms of complete elliptic integrals (Ludwig et al. 2006, 2013; Jardin 2010). We shall provide an explicit expression in section 4.1, as we treat in detail the application of our method to the calculation of the virtual casing principle. At this point, we just highlight the fact that the singularity in the Green’s function when $x = y$ requires the use of specialized quadrature when the target $x$ is located on the surface $\Gamma$, or regularization methods (Freidberg et al. 1976; Merkel 1986; Chance 1997; Landreman & Boozer 2016; Drevlak et al. 2018; Malhotra et al. 2019b). The purpose of this article is to show that for applications in axisymmetric geometries, the Kapur-Rokhlin quadrature scheme is simpler to implement than the known regularization methods used in the magnetic confinement community, and leads to high order convergence.

### 3. The virtual casing principle for toroidally axisymmetric domains

#### 3.1. Formulation of the virtual casing principle

For axisymmetric confinement devices, the virtual casing principle is most often used to compute the poloidal flux or the poloidal magnetic field due to the toroidal current flowing in the plasma (Shafranov & Zakharov 1972; Zakharov 1973; Zakharov & Pletzer 1999; Hirshman & Neilson 1986). A poloidal magnetic field $B^{pol}$ at any point $y(\phi, t) \in \Gamma$ can be expressed in terms of its poloidal flux function $\psi(r, z)$ and the parameterization
Consider an axisymmetric plasma confined by external coils. The poloidal field at any location is the sum of the poloidal field due to the external coils, and of the poloidal field due to the plasma current. We write the latter as \(B_{V}^{\text{pol}}\), and it is given for all \(x \in \mathbb{R}^3\) by the Biot-Savart law:

\[
B_{V}^{\text{pol}}(x) = \frac{\mu_0}{4\pi} \int \int_{\Omega} J_{V}^{\text{tor}}(y) e_{\phi}(y) \times \frac{x - y}{\|x - y\|^3} \, dy
\]  

where \(\mu_0\) is the permeability of free space, and \(J_{V}^{\text{tor}}\) is the toroidal current density in the plasma. Equation (3.2) is a volume integral, which is expensive to evaluate numerically. The virtual casing principle gives a formula for \(B_{V}^{\text{pol}}\) that depends only on the full field \(B^{\text{pol}}\) at the plasma boundary, and only requires the evaluation of a surface integral (i.e. line integral for axisymmetric domains) (Shafranov & Zakharov 1972; Zakharov 1973; Hanson 2015). Specifically, it states that if \(\Gamma\) is the flux surface bounding the plasma, then \(B_{V}^{\text{pol}}\) can be written in terms of a field generated by the toroidal surface current \(J_{S}^{\text{tor}}\) such that \(\mu_0 J_{S}^{\text{tor}} = -n \times B^{\text{pol}}\), according to Hanson (2015):

\[
B_{V}^{\text{pol}}(x) = \frac{1}{4\pi} \int_{\Gamma} \left[ \frac{(n(y) \times B^{\text{pol}}(y)) \times (x - y)}{\|x - y\|^3} \right] \, d\Gamma(y) + \begin{cases} B^{\text{pol}}(x) & x \in \Omega \\ B^{\text{pol}}(x)/2 & x \in \Gamma \\ 0 & x \notin \bar{\Omega} \end{cases}
\]  

(3.3)

For certain applications, one is only interested in the normal component of the poloidal magnetic field (Merkel 1986; Hirshman et al. 1986; Merkel 1987; Landreman 2017; Zhu et al. 2018). The previous equation then leads to the more compact form

\[
n(x) \cdot B_{V}^{\text{pol}}(x) = \frac{1}{4\pi} n(x) \cdot \int_{\Gamma} \left[ \frac{(n(y) \times B^{\text{pol}}(y)) \times (x - y)}{\|x - y\|^3} \right] \, d\Gamma(y)
\]  

(3.4)

for \(x \in \Gamma\).

The reduction of the integral necessary to compute the field or its normal derivative from a volume integral to a surface integral is convenient from the point of view of the limited number of values that need to be specified as inputs, and also from the point of view of the computational cost of the integration (Lazerson et al. 2013). The surface integral in (3.3) is significantly faster to evaluate than the volume integral (3.2), although certain codes still choose to compute the latter (Hanson et al. 2009; Marx & Lütjens 2017).

For axisymmetric situations, one may further take advantage of the axisymmetry of \(B^{\text{pol}}\) to integrate the integral with respect to \(\phi\) analytically, and reduce (3.3) and (3.4) to one-dimensional integrals, which are even less computationally expensive. However, one encounters a mathematical and computational difficulty if one does so, because the surface integrals in (3.3) and (3.4) are in fact improper integrals, which must be understood in the Cauchy principal value sense. This is what we discuss in the following section.
3.2. Numerical evaluation of the normal component of the virtual-casing magnetic field in axisymmetric systems

3.2.1. Circumventing integrals in the principal value sense

Most applications in magnetic confinement fusion rely on version (3.4) of the virtual casing principle, in which one wants to compute the normal component of the poloidal magnetic field \( B_{V}^{pol} \) at a boundary point \( x \in \Gamma \). One could in principle calculate this normal component by first computing all the components of \( B_{V}^{pol} \) by the virtual casing principle (3.3), and then computing \( n \cdot B_{V}^{pol} \) by a straightforward inner product. In other words, one first evaluates the double integral in (3.4), and then takes the dot product with the normal vector \( n \) to the surface \( \Gamma \) at the point \( x \) of interest. This method has the advantage that it produces all components of the poloidal magnetic field \( B_{V}^{pol} \) as intermediary results. However, it also has the disadvantage that one must use a careful principal value integration procedure to interpret the virtual casing principle for \( x \in \Gamma \), as discussed in theorem 2 and its proof in the appendix. The high-order singularity cancellation quadrature scheme we recently proposed for singular integrals on general non-axisymmetric surfaces (Malhotra et al. 2019b) automatically yields the appropriate principal value of the integral. However, it does so thanks to the intrinsic two-dimensional nature of the integral. It does not reduce to a simple and efficient one-dimensional quadrature scheme for the principal value of the integral, as is needed for axisymmetric applications. Similarly, we are not aware of a version of the Kapur-Rokhlin quadrature scheme designed to calculate the Cauchy principal value of the virtual-casing integral.

To address this difficulty, we consider an alternative method to compute the normal poloidal field \( n \cdot B_{V}^{pol} \), based on calculating the vector potential \( A_{S} \) produced by the surface current \( J_{tor}^{S} \), and then obtaining \( n \cdot B_{V}^{pol} \) as the tangential derivative of the poloidal flux \( \psi_{S} \) caused by the surface currents, which is easily expressed in terms of \( A_{S} \). This method is also used, in a slightly different form, by the stellarator optimization code ROSE (Drevlak et al. 2018), but ROSE does not combine it with a high order quadrature scheme. The vector potential is defined by

\[
A_{S}(x) = \frac{\mu_{0}}{4\pi} \int_{\Gamma} \frac{J_{tor}^{S}(y)}{\|x - y\|} d\Gamma(y).
\]

and we recognize this expression as a vector of component-wise single layer potentials for Laplace’s equation in three dimensions, as introduced in Section 2.2.

In contrast to the first approach, our method does not produce all components of \( B_{V}^{pol} \). On the other hand, it only requires weakly singular integrals, since it only requires evaluations of single-layer potentials in (3.5), as we will prove in Section 4. In the next section, we describe in detail this alternative method for computing the normal magnetic field.

3.2.2. Normal component of the magnetic field as a tangential derivative

As in the derivation of the virtual-casing principle (Shafranov & Zakharov 1972; Zakharov 1973; Hanson 2015), one can interpret the plasma boundary \( \Gamma \) of the axisymmetric plasma as a perfectly conducting shell which contributes to confining the plasma via the poloidal magnetic field \( B_{S}^{pol} \) generated by the toroidal surface current density \( J_{S}^{tor} \) flowing in \( \Gamma \). The decomposition \( B_{S}^{pol} = B_{V}^{pol} + B_{S}^{pol} \) then immediately yields \( n \cdot B_{V}^{pol} = -n \cdot B_{S}^{pol} \).

Now, recall that we may represent \( B_{S}^{pol} \) in axisymmetry via (3.1) to express its normal
component at \((r, \phi, z)\) as
\[
\mathbf{n} \cdot \mathbf{B}^{\text{pol}}_S = \mathbf{n} \cdot (\nabla \psi_S \times \nabla \phi) = -\frac{1}{r} (\mathbf{n} \times \mathbf{e}_\phi(\phi)) \cdot \nabla \psi_S
\]
by the circular shift identity of the vector triple product. Let \(x\) correspond to the parameter pair \((\phi_0, t_0)\) with \(x = (R, \phi_0, Z) = (r(t_0), \phi_0, z(t_0))\). We define the tangent vector \(t\) at a point \(x \in \Gamma\) as
\[
t(x) = \mathbf{n}(x) \times \mathbf{e}_\phi(\phi_0)
\]
\[
= \frac{(z'(t_0)e_r(\phi_0) - r'(t_0)e_z) \times e_\phi(\phi_0)}{\sqrt{r'(t_0)^2 + z'(t_0)^2}}
\]
\[
= \frac{r'(t_0)e_r(\phi_0) + z'(t_0)e_z}{\sqrt{r'(t_0)^2 + z'(t_0)^2}}.
\]
It follows that
\[
\mathbf{n}(x) \cdot \mathbf{B}^{\text{pol}}_V(x) = \frac{1}{r(t_0)} t(x) \cdot \nabla \psi_S(r(t_0), z(t_0))
\]
\[
= \frac{1}{r(t_0)\sqrt{r'(t_0)^2 + z'(t_0)^2}} \left( \frac{\partial \psi_S}{\partial r} r'(t_0) + \frac{\partial \psi_S}{\partial z} z'(t_0) \right)
\]
\[
= \frac{1}{\mathcal{J}(t_0)} \frac{\partial \psi_S(r(t_0), z(t_0))}{\partial t_0},
\]
where \(\mathcal{J}(t_0) = r(t_0)\sqrt{r'(t_0)^2 + z'(t_0)^2}\) is the Jacobian of the parameterization introduced in Section 2.1.

Finally, the poloidal flux function and the vector potential are related at \(x\) by the relation (Jardin 2010; Freidberg 2014)
\[
\psi_S(R, Z) = R\mathbf{e}_\phi(\phi_0) \cdot \mathbf{A}_S(x). \tag{3.6}
\]

Given only point evaluations of \(\mathbf{A}_S\) at equispaced parameters \(\{t_i\}\), one can compute \(\partial \psi_S/\partial t\) with high order accuracy at each \(t_i\) by Fourier differentiation. This leads to our high order accurate approach to the virtual casing principle in axisymmetric geometries. One computes \(\mathbf{n} \cdot \mathbf{B}^{\text{pol}}_V\) on \(\Gamma\) by evaluating a weakly singular integral with a high order accurate quadrature rule, and then Fourier differentiating. We show in Section 5 that Kapur-Rokhlin quadrature provides a simple way to obtain high order accuracy for the weakly singular integral. Before we do so, we prove in the next section that it indeed is a weakly singular integral, whose properties satisfy the requirements to obtain high accuracy with the Kapur-Rokhlin quadrature rule.

4. Analytical Results for Singular Integrals

4.1. Vector potential singularity

By the virtual casing principle, we have \(\mu_0 \mathbf{J}_{\text{tor}}^S = -\mathbf{n} \times \mathbf{B}^{\text{pol}}\), so we may equivalently write the vector potential as
\[
\mathbf{A}_S(x) = -\frac{1}{4\pi} \int_{\Gamma} \int \frac{\mathbf{n}(y) \times \mathbf{B}^{\text{pol}}_S(y)}{||x - y||} \mathbf{d}\Gamma(y), \tag{4.1}
\]
which satisfies \(\mathbf{B}^{\text{pol}}_S = \nabla \times \mathbf{A}_S\). As mentioned earlier, we may view this expression as a vector of component-wise single layer potentials for Laplace’s equation in three dimensions. Physically, it is clear that the only non-zero component of the vector
potential is in the $e_\phi$ direction. Furthermore, Chance (1997, §V) shows that after integrating in the toroidal angle $\phi$, the one-dimensional integrands of both the single layer potential $S[\mu_0 J_{S}^{\omega}]$ and the double layer potential $D[\mu_0 J_{S}^{\omega}]$ have integrable, logarithmic singularities. Chance shows this in a modified coordinate system using the poloidal flux function as a coordinate. Next, we will verify these results for the virtual casing principle in standard cylindrical coordinates.

### 4.2. Analytic Reduction to a Line Integral

We shall analytically simplify the integral in (4.1) by integrating over the toroidal angle. When we do so, we shall introduce the complete elliptic integrals of the first and second kind, which are defined as

$$K(k^2) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{and} \quad E(k^2) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta,$$

respectively. Our main analytical result gives a formula for the vector potential of the virtual casing principle as a one-dimensional integral.

**Theorem 1.** Consider the vector potential

$$A_S(x) = -\frac{1}{4\pi} \int \int_{\Gamma} \frac{n(y) \times B_{pol}(y)}{||x - y||} d\Gamma(y)$$

for $x = (R, \phi_0, Z) = (r(t_0), \phi_0, z(t_0))$ in cylindrical coordinates. Define the quantities

$$\alpha = \alpha(t; x) = r(t)^2 + R^2 + (Z - z(t))^2 \quad \text{and} \quad \beta = \beta(t; x) = 2Rr(t)$$

and set the modulus

$$k^2 = k(t; x)^2 = \frac{2\beta}{\alpha + \beta} = \frac{4Rr(t)}{(R + r(t))^2 + (Z - z(t))^2}.$$

Then the vector potential can be expressed as

$$A_S(x) = -e_{\phi}(\phi_0) \int_0^L A(t; x) dt \quad (4.2)$$

with the scalar-valued integrand

$$A(t; x) = \frac{1}{4\pi} \left( \frac{4}{\sqrt{\alpha + \beta}} \right) \left[ \frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t) \right] \left( \frac{2}{k^2} (K(k^2) - E(k^2)) - K(k^2) \right).$$

**Proof.** The unit outward normal vector of $\Gamma$ at $y$ is given (by our assumption on the orientation of $\gamma$) by

$$n(y(\phi, t)) = \frac{\partial y}{\partial \phi} \times \frac{\partial y}{\partial t} \perp \left\| \frac{\partial y}{\partial \phi} \times \frac{\partial y}{\partial t} \right\| = \frac{z'(t)e_r(\phi) - r'(t)e_z}{\sqrt{r'(t)^2 + z'(t)^2}}.$$

From the representation (3.1) of $B_{pol}$ in axisymmetry, it follows that

$$(n(y(\phi, t)) \times B_{pol}(y(\phi, t)))J(t) = \left[ \frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t) \right] e_{\phi}(\phi). \quad (4.3)$$
Next, we consider the difference
\[ x - y(\phi, t) = [Re_\phi(\phi_0) + Ze_z] - [r(t)e_\phi(\phi) + z(t)e_z] \]
\[ = (R \cos \phi_0 - r(t) \cos \phi)e_x + (R \sin \phi_0 - r(t) \sin \phi)e_y + (Z - z(t))e_z, \]
which we have expressed in both cylindrical and rectangular coordinates. Recall that the unit vector \( e_z \) is identical in both coordinate systems, and the remaining rectangular unit vectors \( e_x \) and \( e_y \) are related to standard cylindrical unit vectors by
\[ e_r(\phi) = \cos \phi e_x + \sin \phi e_y \quad \text{and} \quad e_\phi(\phi) = -\sin \phi e_x + \cos \phi e_y. \]

From the representation in rectangular coordinates, we use the trigonometric identity
\[ \cos(\phi_0 - \phi) = \cos \phi_0 \cos \phi + \sin \phi_0 \sin \phi \]
and immediately obtain
\[ \|x - y(\theta, t)\|^2 = R^2 + r(t)^2 + (Z - z(t))^2 - 2Rr(t) \cos(\phi_0 - \phi) = \alpha - \beta \cos(\phi_0 - \phi). \quad (4.4) \]

We have now shown that the surface integral (4.1) for the vector potential is equivalent to the following double integral over a rectangle in the parameter domain:
\[ A_s(x) = -\frac{1}{4\pi} \int_0^L \left[ \frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t) \right] \int_0^{2\pi} \frac{e_\phi(\phi)}{\sqrt{\alpha(t) - \beta(t) \cos(\phi_0 - \phi)}} \, d\phi \, dt. \]

We may analytically evaluate the inner integral using trigonometric identities and known integral formulae. We use the identities
\[ \begin{cases} 
\sin(\phi + \phi_0) = \sin \phi \cos \phi_0 + \cos \phi \sin \phi_0 \\
\cos(\phi + \phi_0) = \cos \phi \cos \phi_0 - \sin \phi \sin \phi_0 
\end{cases} \]
to compute that
\[ \int_0^{2\pi} \frac{-\sin \phi e_x + \cos \phi e_y}{\sqrt{\alpha - \beta \cos(\phi_0 - \phi)}} \, d\phi = \int_0^{2\pi} \frac{-\sin(\phi + \phi_0) e_x + \cos(\phi + \phi_0) e_y}{\sqrt{\alpha - \beta \cos \phi}} \, d\phi 
= e_\phi(\phi_0) \int_0^{2\pi} \frac{\cos \phi \, d\phi}{\sqrt{\alpha - \beta \cos \phi}} - e_r(\phi_0) \int_0^{2\pi} \frac{\sin \phi \, d\phi}{\sqrt{\alpha - \beta \cos \phi}} 
= e_\phi(\phi_0) \int_0^{2\pi} \frac{\cos \phi \, d\phi}{\sqrt{\alpha - \beta \cos \phi}}. \]

Here, we have used the fact that
\[ \int_0^{2\pi} \frac{\sin \phi \, d\phi}{\sqrt{\alpha - \beta \cos \phi}} = 0, \]
which holds because it is the integral of an odd function over a single period.

Now, the remaining integral can be expressed in terms of the complete elliptic integrals,
through the identity
\[
\int_0^{2\pi} \frac{\cos \phi \, d\phi}{\sqrt{\alpha - \beta \cos \phi}} = 2 \int_{-\pi/2}^{\pi/2} \frac{\cos(2\phi + \pi) \, d\phi}{\sqrt{\alpha - \beta \cos(2\phi + \pi)}}
\]
\[
= 2 \int_{-\pi/2}^{\pi/2} \frac{- (1 - 2 \sin^2 \phi) \, d\phi}{\sqrt{\alpha + \beta (1 - 2 \sin^2 \phi)}}
\]
\[
= \frac{-4}{\sqrt{\alpha + \beta}} \int_0^{\pi/2} \frac{1 - 2 \sin^2 \phi \, d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}
\]
\[
= \frac{-4}{\sqrt{\alpha + \beta}} \left( K(k^2) - \frac{2}{k^2} (K(k^2) - E(k^2)) \right). \quad (4.5)
\]

We obtain the last equality (4.5) from the definition of $K$ and from a formula of Gradshteyn & Ryzhik (2014, 2.584-4). The desired result now immediately follows.

The Kapur-Rokhlin quadrature rule that we will implement applies to integrand with a logarithmic singularity. With the result we just obtained, we can verify, by analyzing the behavior of the complete elliptic integrals, that integrand $A$ is indeed logarithmically singular as $t \to t_0$, in agreement with the results from Chance (1997, §V). Specifically, as the modulus $k$ tends to 1, the second-kind integral $E$ is continuous and bounded, and the first-kind elliptic integral $K$ is logarithmically singular. The mapping $t \mapsto k(t; x)^2$ is continuous, so $E(k(t; x)^2)$ is continuous and $K(k(t; x)^2)$ is logarithmically singular as $t \to t_0$. Readers interested in more detail regarding these results are referred to lemma 2 in the appendix.

5. Kapur-Rokhlin quadrature rules

The Kapur-Rokhlin quadrature rules (Kapur & Rokhlin 1997) are a collection of high-order schemes for computing
\[
\int_0^b f(t) \, dt
\]
when $f$ has an integrable singularity at the origin. We will focus on the specific Kapur-Rokhlin scheme for a logarithmic singularity of the form $f(t) = p(t) \log t + q(t)$, where $p$ and $q$ need not have known formulae, but are assumed sufficiently smooth.

5.1. A quadrature rule for nonperiodic functions

The Kapur-Rokhlin quadrature rules are corrections to the trapezoidal rule. For the standard trapezoidal rule with equal spacing $h = b/M$, the quadrature nodes for a nonsingular integrand would be $t_i = ih$ for $i = 0, \ldots, M$. However, we omit the quadrature node $t_0 = 0$ since the integrand $f$ is singular there. This yields the punctured trapezoidal rule
\[
\int_0^b f(t) \, dt \approx h \left[ \sum_{i=1}^{M-1} f_i + \frac{1}{2} f_M \right]
\]
where we have written the shorthand $f_i = f(t_i)$.

The Kapur-Rokhlin corrections place additional quadrature nodes outside the integration domain $[0, b]$. Specifically, the corrections depend on a convergence rate parameter $n$ and a smoothness parameter $m$. One chooses $m \geq 3$ an odd integer subject to the
constraint that $p$ and $q$ are $m$ times continuously differentiable on a wider interval $[-nh, b + mh]$. Under these conditions, the Kapur-Rokhlin scheme sets constants $\gamma_j$ for $j = \pm 1, \ldots, \pm n$ and $\beta_l$ for $l = 1, \ldots, (m - 1)/2$ and defines the quadrature rule

$$T^{M}_{m,n}(f) = h \left[ \sum_{i=1}^{M-1} f_i + \frac{1}{2} f_M \right] + h \sum_{l=1}^{(m-1)/2} \beta_l (f_{M-l} - f_{M+l}) + h \sum_{1 \leq |j| \leq n} \gamma_j f_j.$$  

for $M \geq n + (m - 1)/2$. The error obeys the asymptotic rate

$$\left| T^{M}_{m,n}(f) - \int_0^b f(t) dt \right| = O(h^n)$$

as $h \to 0$.

5.2. A simplified quadrature for periodic functions

This subsection follows an argument nearly verbatim from Hao et al. (2014). We explain how the Kapur-Rokhlin scheme we just presented simplifies when computing $\int_0^b f(t) dt$ when $f$ is $2b$-periodic and logarithmically singular at the origin. We may express these assumed properties of $f$ through the form

$$f(t) = p(t) \log \left| \sin \frac{\pi t}{2b} \right| + q(t),$$

for $2b$-periodic functions $p$ and $q$.

We sum two applications of the original Kapur-Rokhlin scheme — one for

$$I_1 = \int_0^b f(t) dt$$

and another for

$$I_2 = \int_{-b}^0 f(t) dt = \int_0^b f(-t) dt.$$

We assume that $p, q \in C^m[-b, b]$ and generate $2M - 1$ equispaced quadrature nodes with spacing $h = b/M$, defined by $t_i = ih$ for $i = \pm 1, \ldots, \pm (M - 1), M$. The corrected scheme for $I_1$ is

$$I_1 = h \left[ \sum_{i=1}^{M-1} f_i + \frac{1}{2} f_M \right] + h \sum_{l=1}^{(m-1)/2} \beta_l (f_{M-l} - f_{M+l}) + h \sum_{1 \leq |j| \leq n} \gamma_j f_j + O(h^n),$$

and the scheme for $I_2$ is

$$I_2 = h \left[ \sum_{i=1}^{M-1} f_i + \frac{1}{2} f_M \right] + h \sum_{l=1}^{(m-1)/2} \beta_l (f_{M+l} - f_{M-l}) + h \sum_{1 \leq |j| \leq n} \gamma_j f_{-j} + O(h^n).$$

By periodicity, we may identify $f_i$ with $f_{i+2M}$ for all $i$. It follows that an $n$th order quadrature for $I_1 + I_2$ is

$$I_1 + I_2 = h \left[ \sum_{1 \leq |i| \leq M-1} f_i + f_M \right] + h \sum_{1 \leq |j| \leq n} \gamma_j (f_j + f_{-j}) + O(h^n).$$
We note that, by periodicity, the endpoint corrections corresponding to the constants \( \beta_l \) exactly cancel. Moreover, we no longer need to place additional quadrature nodes beyond \([-b, b]\) because periodicity identifies the new quadrature nodes with existing nodes.

Given a table of \( \gamma_j \) weights, this quadrature scheme is as easy to implement as the trapezoidal rule, and yields high order convergence, with a quadrature error that is \( O(h^n) \). The necessary \( \gamma_j \) weights can be found in Kapur & Rokhlin (1997, table 6) for \( n = 2, 6, 10 \).

### 5.3. A Kapur-Rokhlin scheme for the virtual casing principle

We may compute the vector potential by equation (4.2) of theorem 1 using the Kapur-Rokhlin quadrature rule for periodic functions. Let \( x \in \Gamma \) be given, corresponding to parameters \((\phi_0, t_0)\). We reiterate that a univariate integral expression for the vector potential is

\[
A_S(x) = -e_\phi(\phi_0) \int_0^L A(t; x) dt
\]

The Kapur-Rokhlin quadrature of the previous subsection applies directly because the integration interval \([0, L]\) is identical, by periodicity, to the symmetric interval \([t_0 - L/2, t_0 + L/2]\) and because the integrand \( A(t; x) \) is logarithmically singular as \( t \to t_0 \).

Given \( M \in \mathbb{N} \), we generate \( 2M - 1 \) quadrature nodes \( t_i = t_0 + ih \) for \( i = \pm 1, \ldots, \pm (M - 1) \), \( M \) with spacing \( h = L/(2M) \) as in the previous subsection. We evaluate \( A_i = A(t_i; x) \), and it follows that

\[
\int_0^L A(t; x) dt = h \left[ \sum_{1 \leq |i| \leq M-1} A_i + A_M \right] + h \sum_{1 \leq |j| \leq n} \gamma_j (A_j + A_{-j}) + O(h^n). \tag{5.1}
\]

Since periodicity has removed our considerations about expanding the integration domain, the parameter \( n \) can be taken as large as we like (as long as \( M \geq n \) to define the quadrature rule). However, in practice this Kapur-Rokhlin scheme is known to be unstable for \( n \) larger than about 10 because the weights \( \gamma_j \) are sign-indefinite and grow large in magnitude.

### 6. Numerical results

In this section, we illustrate the Kapur-Rokhlin quadrature schemes from sections 5.2 and 5.3 in two different calculations. Throughout, we will test the schemes for an axisymmetric plasma boundary given by the level set \( \psi = 0 \) of the poloidal flux function given by

\[
\psi(r, z) = \frac{\kappa F_B}{2 R_0^2 q_0} \left[ \frac{1}{4} (r^2 - R_0^2)^2 + \frac{1}{\kappa^2} r^2 z^2 - a^2 R_0^2 \right].
\]

This flux function is a solution to the Grad-Shafranov equation with the Solov'ev profiles \( \mu_0 \psi(\psi) = -[F_B(\kappa + 1/\kappa)/(R_0^2 q_0)]\psi \) and \( F(\psi) = F_B \), where \( p(\psi) \) is the plasma pressure profile, and \( F(\psi) = rB_\phi \), with \( B_\phi \) the toroidal magnetic field (Lütjens et al. 1996; Lee & Cerfon 2015). The parameters \( R_0 \) and \( q_0 \) may be interpreted as the major radius and safety factor at the magnetic axis, and \( \kappa \) and \( a \) as the elongation and minor radius of the plasma boundary. All numerical tests in this manuscript use the fusion relevant values \( F_B = R_0 = q_0 = 1 \) and \( \kappa = 1.7 \) and \( a = 1/3 \). The level set \( \psi = 0 \) may be parameterized...
by the functions (Lütjens et al. 1996; Lee & Cerfon 2015)
\[(r(t))^2 = R_0^2 + 2aR_0 \cos t\quad \text{and} \quad z(t) = \kappa a \frac{R_0}{r(t)} \sin t\]
for \(t \in [0, 2\pi]\).

6.1. Double layer identity

For our first numerical verification, we consider an identity associated with harmonic functions. Consider the Green’s function \(G(x, y) = (4\pi \|x - y\|)^{-1}\) for Laplace’s equation in three dimensions. It satisfies the double layer jump condition (Malhotra et al. 2019b)
\[
\iint_{\Gamma} \frac{\partial G(x, y)}{\partial n(y)} d\Gamma(y) = \frac{1}{4\pi} \iint_{\Gamma} \frac{n(y) \cdot (x - y)}{\|x - y\|^3} d\Gamma(y) = -\frac{1}{2}
\]
for \(x \in \Gamma\). It follows that
\[
1 + \frac{1}{2\pi} \iint_{\Gamma} \frac{n(y) \cdot (x - y)}{\|x - y\|^3} d\Gamma(y) = 0,
\]
again for \(x \in \Gamma\). Following identical methodology to the proof of theorem 1, we integrate out the toroidal angle analytically to obtain the one-dimensional integral identity
\[
1 + \frac{1}{2\pi} \frac{4r}{(\alpha + \beta)^{3/2}} \left\{ -\frac{2z_r R'}{k^2} K(k^2) + \left( \frac{2z_r R}{k^2} + \frac{z_r (R - r) - r'(Z - z)}{1 - k^2} \right) E(k^2) \right\} dt = 0.
\]  
In the above expression, we have suppressed the dependence of \(\{r, z, r', z', \alpha, \beta, k^2\}\) on \(t\) for clarity. As usual, we have also used the identification \(x = (R, \phi_0, Z) = (r(t_0), \phi_0, z(t_0))\). The integrand is logarithmically singular because of the presence of the singular elliptic integral \(K(k^2)\), and because the other seemingly singular coefficient is actually bounded, with
\[
\lim_{t \to t_0} \frac{z_r(t)(R - r(t)) - r'(t)(Z - z(t))}{1 - k(t)^2} = 2R^2 \left( \frac{r''(t_0)z'(t_0) - r'(t_0)z''(t_0)}{r'(t_0)^2 + z'(t_0)^2} \right).
\]
We conclude that the integral in (6.1) is of the Kapur-Rokhlin form from section 5.2.

Figure 1 illustrates the performance of the 10th order periodic Kapur-Rokhlin quadrature scheme for verifying the identity (6.1). We compare this method with the alternating trapezoidal rule, which uses common quadrature weight \(h = \pi/M\) and quadrature nodes \(\tilde{t}_i = t_0 + (i - \frac{1}{2})h\) for \(i = 0, \pm 1, \ldots, \pm (M - 1), M\) that straddle the singularity at \(t_0\). We find that the Kapur-Rokhlin scheme achieves the theoretical 10th order accuracy, and that we obtain the full accuracy of the quadrature scheme using about 175 quadrature nodes. Because the behavior of the integrand is not generally symmetric about \(t = t_0\), the alternating trapezoidal rule performs poorly.

6.2. Virtual casing principle

For our second test, we evaluate the accuracy of our method for calculating the normal component of the magnetic field due to the plasma current on the plasma boundary, i.e. \(n \cdot B_{\text{pol}}^r\) on \(\Gamma\). The plasma equilibrium we consider is the Solov’ev equilibrium described above (Lütjens et al. 1996; Lee & Cerfon 2015). Since we do not know the analytic solution to this problem, we compare our implementation with an existing high order accurate implementation from Malhotra et al. (2019b). The method used in that implementation
is different from the approach presented here in several ways, making it appropriate for our verification. Specifically, the code presented in (Malhotra et al. 2019b) views the plasma equilibrium as a fully three-dimensional equilibrium, and does not assume axisymmetry. Furthermore, Malhotra et al. (2019b) obtain $\mathbf{n} \cdot \mathbf{B}_{V}^{\text{pol}}$ on $\Gamma$ directly via a direct evaluation of (3.3), as opposed to first computing the vector potential. Finally, the Cauchy principal value of (3.3) is numerically evaluated via a partition of unity scheme to handle the singularity of the integrand. We take the result from a high resolution calculation of Malhotra et al. (2019b) for this problem as the ground truth, against which we test the accuracy of our approach as a function of the number of quadrature points.

Figure 2 illustrates our results when using the 10th order Kapur-Rokhlin scheme for this problem. Again, we find that the Kapur-Rokhlin scheme achieves the theoretical convergence rate, and that the accuracy has converged by about 400 quadrature nodes. In both this example and the double layer identity example, we observe that the Kapur-Rokhlin scheme errors do not converge all the way to machine precision. This is one known drawback of the Kapur-Rokhlin methods, driven partially by the instabilities caused by the correction weights. Nonetheless, in contexts where full machine precision is not necessary, this scheme provides high accuracy and is easy to implement by reading the correction weights $\gamma_j$ from a table of pre-computed values.

7. Conclusion

For axisymmetric confinement fusion systems, a direct implementation of the Kapur-Rokhlin quadrature scheme yields high accuracy for the evaluation of the layer potentials commonly encountered in magnetostatic and magnetohydrodynamic calculations. Using the table of quadrature weights given in the original article by Kapur and Rokhlin (Kapur & Rokhlin 1997), the scheme is as easy to implement as the trapezoidal rule, and, unlike commonly used methods, does not require any manipulation of the singular integrands.
We demonstrated how to implement it for the evaluation of a double-layer potential and for the virtual casing principle, obtaining 10th order convergence in both cases.

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Appendix A

In this appendix, we prove that the virtual casing principle (3.3) is undefined by standard integration when the evaluation point lies on the boundary. In order to prove this, we require the following two lemmas concerning the asymptotic behavior of the complete elliptic integral $K(k(t)^2)$ for values $t$ close to $t_0$. Throughout, we will identify the evaluation point $x \in \Gamma$ with the parameters $(\phi_0, t_0)$, so that $x = (r(t_0), \phi_0, z(t_0)) = (R, \phi_0, Z)$.

Lemma 1. Assume $r(t)$ and $z(t)$ are $L$-periodic $C^1[0, L]$ parameterization functions. Define the modulus by

$$k(t)^2 = \frac{4Rr(t)}{(R + r(t))^2 + (Z - z(t))^2}.$$  

Let $|t - t_0| > 0$ be sufficiently small. Then

$$k(t)^2 = \left(1 + (t - t_0)^2 \left(\frac{r'(s_r)^2 + z'(s_z)^2}{4R(r(t)^2 + z(t)^2)}\right)\right)^{-1}$$

and

$$1 - k(t)^2 = \left(1 + (t - t_0)^2 \left(\frac{4Rr(t)^2 + z(t)^2}{r'(s_r)^2 + z'(s_z)^2}\right)\right)^{-1}$$

for some $s_r, s_z$ between $t_0$ and $t$.

Proof. We use the Taylor expansions

$$r(t) = r(t_0) + (t - t_0)r'(s_r) = R + (t - t_0)r'(s_r)$$
$$z(t) = z(t_0) + (t - t_0)z'(s_z) = Z + (t - t_0)z'(s_z)$$

for some $s_r, s_z$ between $t_0$ and $t$. Inserting this into the modulus, we obtain

$$k(t)^2 = \frac{4R(r(t)^2 + z(t)^2)}{4R(R + (t - t_0)r'(s_r)) + (t - t_0)^2r'(s_r)^2 + (t - t_0)^2z'(s_z)^2}$$

$$= \left(\frac{4R(R + (t - t_0)r'(s_r)) + (t - t_0)^2(r'(s_r)^2 + z'(s_z)^2)}{4R + (t - t_0)r'(s_r)}\right)^{-1}$$

Simplifying the fraction yields the desired expression for $k(t)^2$. It is quick to verify the expression for $1 - k(t)^2$ accordingly.

Lemma 2. Assuming the same conditions as lemma 1, the complete elliptic integral of the first kind obeys the asymptotic behavior

$$K(k(t)^2) = -\log |t - t_0| + O(1) \quad \text{as } t \to t_0.$$  

Proof. We first observe that $k(t)^2 \to 1^-$ as $t \to t_0$. In this regime, Gradshteyn & Ryzhik (2014, 8.113-3) gives the asymptotic expansion

$$K(k^2) = \log \left(\frac{4}{\sqrt{1 - k^2}}\right) + o(1) \quad \text{as } k^2 \to 1^-.$$
From lemma 1, we also have the explicit representation of the dominant term

$$\log \left( \frac{4}{\sqrt{1-k^2}} \right) = \log 4 + \frac{1}{2} \log \left( 1 + |t-t_0|^{-2} \left( \frac{4R(R + (t-t_0)r'(s_r))}{r'(s_r)^2 + z'(s_z)^2} \right) \right).$$

Finally, we use the identity \(\log(1+u) = \log u + \log(u^{-1} + 1) = \log u + o(1)\) as \(u \to \infty\). We conclude that

$$K(k(t)^2) = \log 4 + \frac{1}{2} \log \left( |t-t_0|^{-2} \left( \frac{4R(R + (t-t_0)r'(s_r))}{r'(s_r)^2 + z'(s_z)^2} \right) \right) + o(1)$$

$$= \log 4 - \log |t-t_0| + \frac{1}{2} \log \left( \frac{4R(R + (t-t_0)r'(s_r))}{r'(s_r)^2 + z'(s_z)^2} \right) + o(1)$$

as \(t \to t_0\). Only the second term is singular as \(t \to t_0\); the others are bounded, and this completes the proof.

We now have the necessary tools to prove that the integrand obtained by the virtual casing principle is not absolutely integrable on the parameter domain.

**Theorem 2.** Let \(\Gamma\) be a smooth surface of revolution, and let \(x \in \Gamma\). Assume that there exists a constant \(R_{\min} > 0\) such that \(r(t) \geq R_{\min}\) for all \(t \in [0, L]\). Then

$$\iint_{\Gamma} \left| \frac{(n(y) \times B^{pol}(y)) \times (x-y)}{\|x-y\|^3} \right| \, d\Gamma(y) = \infty.$$

**Proof.** Without loss of generality, we assume that the coordinate system is appropriately rotated so that \(x\) has zero toroidal angle. That is, we assume \(x = (R, 0, Z) = (r(t_0), 0, z(t_0))\). In this form, we recall that \(e_r(0) = e_x\) and \(e_\phi(0) = e_y\). The surface integral can be rewritten in the parameter domain as

$$\iint_{\Gamma} \left| \frac{(n(y) \times B^{pol}(y)) \times (x-y)}{\|x-y\|^3} \right| \, d\Gamma(y) = \int_0^L \int_0^{2\pi} \|F(\phi, t)\| \, d\phi \, dt,$$

where \(F\) is expressible from the parameterization representations (4.3) and (4.4) as

$$F(\phi, t) = \frac{\left[ \frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t) \right] e_\phi(\phi) \times \left[ (R e_x + Z e_z) - (r(t)e_r(\phi) + z(t)e_z) \right]}{(\alpha(t) - \beta(t) \cos \phi)^{3/2}}$$

$$= \frac{\left[ \frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t) \right]}{(\alpha(t) - \beta(t) \cos \phi)^{3/2}} \left( (Z - z(t)) e_r(\phi) + (r(t) - R \cos \phi) e_z \right).$$

By Tonelli’s theorem, we may evaluate the integral of \(\|F\|\) in \(\phi\) first, and show that the remaining integral in \(t\) diverges. The integrands in \(\phi\) can be transformed into expressions with formulae known from Gradshteyn & Ryzhik (2014, 2.584-37 and 2.584-42), by a process identical to how we obtained (4.5) in the proof of theorem 1. The result is the
univariate, vector-valued function

\[ F_1(t) = \int_0^{2\pi} F(\phi, t) d\phi \]

\[ = \frac{2}{r(t)\sqrt{\alpha + \beta}} \left\{ \frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t) \right\} \left( -K(k^2) + \frac{\alpha}{\alpha - \beta} E(k^2) \right) e_x \]

\[ + \left( K(k^2) + \frac{r(t)^2 - R^2 - (Z - z(t))^2}{\alpha - \beta} E(k^2) \right) e_z \].

As before, we have introduced the quantities

\[
\begin{align*}
\alpha &= \alpha(t; x) = R^2 + r(t)^2 + (Z - z(t))^2 \\
\beta &= \beta(t; x) = 2Rr(t) \\
k^2 &= k(t; x)^2 = \frac{2\beta}{\alpha + \beta}.
\end{align*}
\]

By the immediate comparison \( \int_0^{2\pi} \| F(\phi, t) \| d\phi \geq \| F_1(t) \| \), it is sufficient to prove our claim by showing that \( \| F_1(t) \| \) is not integrable. We will show that a singularity in one of the components of \( F_1(t) \) must be at least as severe as \( |t - t_0|^{-1} \) as \( t \to 0 \), and this will prove that \( \| F(\theta, t) \| \) is not integrable.

First, we consider integrating \( e_z \cdot F_1(t) \) with the purely formal expression

\[ \int_0^{2\pi} 2(Z - z(t)) \left[ \frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t) \right] \left( -K(k^2) + \frac{\alpha}{\alpha - \beta} E(k^2) \right) dt. \]

For any geometry where \( r(t) \geq R_{\text{min}} > 0 \), and for generic functions \( \psi(r, z) \), the quantity

\[ \lim_{t \to t_0} 2 \left[ \frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t) \right] = \frac{1}{R^3} \left[ \frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t) \right]_{t=t_0} \]

is finite. Moreover, it is also nonzero since for any valid parameterization, the derivatives \( r'(t) \) and \( z'(t) \) cannot concurrently vanish for any fixed \( t \), including \( t = t_0 \). So, the behavior of the singularity in the integrand \( e_z \cdot F_1(t) \) depends purely on what remains.

The first term \( (Z - z(t)) K(k^2) \) is clearly integrable, since \( |Z - z(t)| = O(|t - t_0|) \) and \( K(k(t)^2) = -\log |t - t_0| + O(1) \) as \( t \to t_0 \) by lemma 2. For the second term, we observe that the limit

\[ \lim_{t \to t_0} \frac{\alpha}{\alpha - \beta} E(k^2) = 2R^2 E(1) = 2R^2 \]

is again finite and nonzero, and so we are left to question: How severe is the singularity of

\[ \frac{Z - z(t)}{\alpha - \beta} = \frac{Z - z(t)}{(R - r(t))^2 + (Z - z(t))^2} \]

as \( t \to t_0 \)? Since the parameterizations \( r(t) \) and \( z(t) \) are continuously differentiable functions, we may write Taylor expansions

\[ r(t) = r(t_0) + (t - t_0)r'(s_r) = R + (t - t_0)r'(s_r) \]
\[ z(t) = z(t_0) + (t - t_0)z'(s_z) = Z + (t - t_0)z'(s_z) \]

for some values \( s_r, s_z \) between \( t_0 \) and \( t \), which depend on \( t \) and which tend to \( t_0 \) as \( t \to t_0 \).
It immediately follows that, for fixed $t$, we have
\[
\frac{Z - z(t)}{(R - r(t))^2 + (Z - z(t))^2} = \frac{-(t - t_0)z'(s_z)}{(t - t_0)^2(r'(s_r)^2 + z'(s_z)^2)} = \left(\frac{1}{t - t_0}\right) \left(\frac{-z'(s_z)}{r'(s_r)^2 + z'(s_z)^2}\right).
\]
As long as $z'(t_0) \neq 0$, we obtain the answer to our question. The integrand obeys the asymptotic estimate
\[
e_x \cdot F_1(t) \sim \frac{1}{t - t_0} \quad \text{as } t \to t_0,
\]
and we conclude that $\|F_1(t)\|$, and hence $\|F(\theta, t)\|$, are not integrable.

When $z'(t_0) = 0$, we consider the integral of the other component $e_x \cdot F_1(t)$ and consider whether
\[
\int_0^L 2 \left[\frac{\partial \psi}{\partial z} r'(t) - \frac{\partial \psi}{\partial r} z'(t)\right] \frac{r(t)^2 - R^2 - (Z - z(t))^2}{\alpha - \beta} E(k^2) \, dt
\]
diverges. Using an argument verbatim to the first part of this proof, we conclude that its behavior is determined by the singularity of
\[
\frac{r(t)^2 - R^2 - (Z - z(t))^2}{\alpha - \beta} = \frac{r(t)^2 - R^2 - (Z - z(t))^2}{(R - r(t))^2 + (Z - z(t))^2}.
\]
Using the same Taylor expansions, we obtain
\[
\frac{r(t)^2 - R^2 - (Z - z(t))^2}{(R - r(t))^2 + (Z - z(t))^2} = \frac{2R(t - t_0)r'(s_r) + (t - t_0)^2(r'(s_r)^2 + z'(s_z)^2)}{(t - t_0)^2(r'(s_r)^2 + z'(s_z)^2)}
\]
\[
= \left(\frac{1}{t - t_0}\right) \frac{2Rr'(s_r)}{r'(s_r)^2 + z'(s_z)^2} + \frac{r'(s_r)^2 - z'(s_z)^2}{r'(s_r)^2 + z'(s_z)^2}.
\]
With identical reasoning, and considering that $r'$ and $z'$ cannot simultaneously vanish, we conclude once again that $\|F(\theta, t)\|$ is not integrable. \hfill \square

As a result of theorem 2, we conclude that one must use a principal value procedure in order to define all components of $B_{\text{pol}}^\text{pol}(x)$ by the virtual casing principle (3.3) when $x \in \Gamma$. One can prove that this is possible, i.e., that a limiting procedure yields a finite result, but we omit the lengthy details here.