1 The Black-Scholes Formula for a European Call or Put

Recall:

\[ V(f) = e^{-r(T-t)} \mathbb{E}_{RN}[f(S_T)] \]

where the expectation is taken with respect to the risk-neutral measure.

In a risk-neutral world, the stock price dynamics is

\[ S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Z}, \quad Z \sim \mathcal{N}(0, 1) \]

or equivalently

\[ \log \left( \frac{S_T}{S_t} \right) \sim \mathcal{N}\left( \left( r - \frac{1}{2}\sigma^2 \right)(T-t), \sigma^2(T-t) \right) \]

Note that \( f(S_T) \) is the payoff, a known function of \( S_T \), e.g.,

1.1 Evaluation of European Options

Evaluation of a European Call/Put at \( t = 0 \). Let us quote the results first:

\[ c[S_0, T, K] = S_0 N(d_1) - Ke^{-rT} N(d_2), \]
\[ p[S_0, T, K] = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \]

where

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \]

\[ d_1 = \frac{1}{\sqrt{\sigma^2 T}} \log \left( \frac{S_0 e^{(r+\frac{1}{2}\sigma^2) T}}{K} \right) \]
\[ d_2 = \frac{1}{\sqrt{\sigma^2 T}} \log \left( \frac{S_0 e^{(r-\frac{1}{2}\sigma^2) T}}{K} \right) \]
Note that
\[ d_2 = d_1 - \sqrt{\sigma^2 T} \]

First, let us evaluate the expectation of the following function
\[
f(x) \equiv \begin{cases} e^{ax}, & x \geq k \\ 0, & \text{otherwise} \end{cases}
\]
where \( X \) is a Gaussian-distributed random variable with mean \( m \) and variance \( \sigma^2 \):
\[
E[f(x)] = \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx
\]
\[
= \frac{1}{\sqrt{2\pi \sigma^2}} \int_k^\infty e^{ax} e^{-\frac{(x-m)^2}{2\sigma^2}} dx
\]
Complete the square:
\[
ax - \frac{(x-m)^2}{2\sigma^2} = am + \frac{1}{2} a^2 \sigma^2 - \frac{[x - (m + a\sigma^2)]^2}{2\sigma^2}
\]
therefore,
\[
E[f(x)] = e^{am + \frac{1}{2} a^2 \sigma^2} \frac{1}{\sigma' \sqrt{2\pi}} \int_k^\infty e^{-\frac{[x - (m + a\sigma^2)]^2}{2\sigma'^2}} dx
\]
Changing variable,
\[
y \equiv \frac{x - (m + a\sigma^2)}{\sigma'},
\]
yields
\[
E[f(x)] = e^{am + \frac{1}{2} a^2 \sigma^2} \int_{\frac{k - (m + a\sigma^2)}{\sigma'}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\]
\[
= e^{am + \frac{1}{2} a^2 \sigma^2} \int_{-\infty}^{-\frac{k - (m + a\sigma^2)}{\sigma'}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \quad \text{(even-symmetry of a Gaussian)}
\]
\[
= e^{am + \frac{1}{2} a^2 \sigma^2} N\left(-\frac{k - (m + a\sigma^2)}{\sigma'}\right)
\]

Therefore, we have
\[
E[f(x)] = e^{am + \frac{1}{2} a^2 \sigma^2} N\left(d\right), \quad d \equiv \frac{-k + m + a\sigma^2}{\sigma'} \quad (1)
\]
1.1.1 European Call

Applying Eq. (1) to a European call:

\[ V (f) = e^{rT} \int_{-\infty}^{\infty} (S_0 e^x - K) + \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(x-(r-\frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}} dx \]

Note that

\[ S_0 e^x - K \geq 0 \implies x > \log \frac{K}{S_0} \]

1. For the first term in the payoff, i.e., \( S_0 e^x \), we use the result above with

\[ a = 1, \quad k = \log \frac{K}{S_0}, \quad m = \left( r - \frac{1}{2} \sigma^2 \right) T, \quad \sigma'^2 = \sigma^2 T \]

therefore,

\[ e^{-rT} \int_{k}^{\infty} S_0 e^x \frac{1}{\sqrt{2\pi\sigma'^2 T}} e^{-\frac{(x-(r-\frac{1}{2}\sigma'^2)T)^2}{2\sigma'^2 T}} dx = S_0 e^{-rT} e^{(r-\frac{1}{2}\sigma^2)^T + \frac{1}{2} \sigma^2 T} N (d_1) = S_0 N (d_1) \]

where

\[ d_1 = \frac{-\log \frac{K}{S_0} + (r - \frac{1}{2} \sigma^2) T + \sigma^2 T}{\sqrt{\sigma^2 T}} \]

\[ = \frac{1}{\sqrt{\sigma^2 T}} \log \left[ \frac{S_0 e^{(r+\frac{1}{2}\sigma^2)T}}{K} \right] \]

2. For the 2nd term (i.e., -K), choose

\[ a = 0, \]

then,

\[ e^{-rT} \int_{k}^{\infty} K \frac{1}{\sqrt{2\pi\sigma'^2 T}} e^{-\frac{(x-(r-\frac{1}{2}\sigma'^2)T)^2}{2\sigma'^2 T}} dx = K e^{-rT} N (d_2) \]

where

\[ d_2 = \frac{-\log \frac{K}{S_0} + (r - \frac{1}{2} \sigma^2) T}{\sqrt{\sigma^2 T}} \]

\[ = \frac{1}{\sqrt{\sigma^2 T}} \log \left[ \frac{S_0 e^{(r+\frac{1}{2}\sigma^2)T}}{K} \right] \]

Therefore,

\[ c (S_0, T, K) = S_0 N (d_1) - K e^{-rT} N (d_2) \]
1.1.2 European Put

How to evaluate a put? Use the put-call parity

\[ p - c = Ke^{-rT} - S_0 \]

Therefore,

\[
p(S_0, T, K) = c(S_0, T, K) + Ke^{-rT} - S_0 \\
= S_0 N(d_1) - S_0 + K e^{-rT} - K e^{-rT} N(d_2) \\
= -S_0 (1 - N(d_1)) + K e^{-rT} (1 - N(d_2)) \\
= -S_0 N(-d_1) + K e^{-rT} N(-d_2)
\]

Hence

\[ p(S_0, T, K) = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \]

Note that the notation \( T \) can be understood as the maturity of the contract counting from the day when the option is setup or it can also be understood as the time-to-maturity — which is, sometimes, emphasized through the notation \( T - t \) with \( T \) being reserved for maturity.

Note that

1. These prices are good as long as the lognormal stock price dynamics is a good model for our market;

2. Parameters in the formula:

\( S_0 \) — the present value of a stock,  
\( K \) — Strike,  
\( r \) — risk-free interest rate,  
\( T \) — maturity or time-to-maturity  
\( \sigma \) — volatility

what is the value of \( \sigma \)?
(a) Historical volatility;  
(b) Implied volatility — cf. Volatility smile, volatility skew.

Issues:

1. How good is the lognormal dynamics?  
2. How to hedge away some of the problem?

2 Hedging

2.1 Hedging in a Binomial World

\[ f_0 = f (S_0, t + \delta t) + \left[ -rf(S_0, t + \delta t) + rS_0 \frac{\partial f}{\partial S_0} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 f}{\partial S_0^2} \right] \delta t + o(\delta t) \]

where all the derivatives are evaluated at \( t + \delta t \).

Suppose we misparameterized \( \sigma \):

\[ \sigma' = \sigma + \delta \sigma \]

where \( \delta \sigma \) is the error.
Then, the incorrect price for our contingent claim:

\[
f'_0 = f(S_0, t + \delta t) + \frac{\partial f}{\partial \sigma} \delta \sigma + \left[-rf(S_0, t + \delta t) + rS_0 \frac{\partial f}{\partial S_0} + \frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 f}{\partial S_0^2}\right] \delta t + o(\delta t)
\]

For the purpose of illustrating the idea of hedging, we have assumed \( \delta \sigma \) and \( \delta t \) are of the same order, otherwise, there are further expansions of those derivatives with respect to \( \delta \sigma \). Here, we neglect higher order terms, e.g. \( O(\delta t \delta \sigma) \).

Therefore, the mispriced amount is

\[
\delta f_0 \equiv f'_0 - f_0 \\
\approx \frac{\partial f}{\partial \sigma} \delta \sigma
\]

However, if we have another contingent claim on the same stock to form a portfolio:

\[ f + x g, \quad x \text{ — the number of unit of } g\text{-option.} \]

Then the total mispricing will be

\[
\delta (f + xg) \approx \left(\frac{\partial f}{\partial \sigma} + x \frac{\partial g}{\partial \sigma}\right) \delta \sigma
\]

if

\[
\frac{\partial f}{\partial \sigma} + x \frac{\partial g}{\partial \sigma} = 0
\]

i.e.,

\[
x = \frac{\frac{\partial f}{\partial \sigma}}{\frac{\partial g}{\partial \sigma}}
\]

then we can hedge away potential mispricing due to incorrect volatility parameterization to \( O(\delta \sigma) \) — a Vega hedging.

Terminology:

\[
\text{Vega : } V_f \equiv \frac{\partial f}{\partial \sigma}, \quad V_g \equiv \frac{\partial g}{\partial \sigma}
\]

i.e.,

\[
x = \frac{V_f}{V_g}
\]

Question: Have we hedged away all risks yet? Let’s analyze this issue further.

Recall that the replicating portfolio in a risk-neutral valuation is

\[
-f_0 + \Delta f S_0 + B_f = 0
\]

replicating portfolio
in a correctly parameterized world. i.e.,
\[-f_+ + \Delta f S_+ + B_f e^{r \delta t} = -f_- + \Delta f S_- + B_f e^{r \delta t}\]
\[\therefore \Delta f = \frac{f_+ - f_-}{S_+ - S_-}\]

is the amount of stock needed to hedge away the risk

Due to \(\sigma\)-misparameterization:
\[-f_0' + \Delta f_0 S_0 + B'_f = 0 \quad \text{at } t = 0\]

therefore,
\[\Delta f = \frac{f'_+ - f'_-}{S_+ - S_-}\]

where \(f'_+, f'_-\) and \(B'_f\) are computed using \(\sigma'\). So in a time-step \(\delta t\), our risk is

\[\delta \Pi_f = (-f_+ + \Delta f_+ S_+ + B'_f e^{r \delta t}) - (-f_- + \Delta f_- S_- + B'_f e^{r \delta t})\]

\[\uparrow \quad \text{N.B. in the real world, our } f \text{ has to pay } f_+ \text{ rather than } f'_+\]

\[\therefore \delta \Pi_f = -(f_+ - f_-) + \Delta f (S_+ - S_-)\]

\[\therefore \Delta f = \Delta f(\sigma)\]

\[\therefore \Delta f' \approx \Delta f + \frac{\partial \Delta f}{\partial \sigma} \delta \sigma + o(\delta \sigma)\]

therefore,
\[\delta \Pi_f = \underbrace{-(f_+ - f_-) + \Delta f (S_+ - S_-)}_{= 0} + \frac{\partial \Delta f}{\partial \sigma} \delta \sigma (S_+ - S_-)\]

\[\therefore \delta \Pi_f \approx \frac{\partial \Delta f}{\partial \sigma} (S_+ - S_-) \delta \sigma\]

which, in general, is not zero. However,
\[S_+ - S_- \approx O\left(\sigma \sqrt{\delta t}\right)\]

\[\therefore \delta \Pi_f = O\left(\frac{\partial \Delta f}{\partial \sigma} \sigma \sqrt{\delta t} \delta \sigma\right)\]

which contains risks — Either we are content to live with these risks (they could be small or large, depending on the combination of \(\frac{\partial \Delta f}{\partial \sigma} \sigma \delta \sigma\)) or we can try to hedge further — Let’s see how theoretically this can be done. First note that, even for our portfolio
\[f + x g\]
we have
\[ \delta \Pi_{f+g} = \delta \Pi_f + \delta \Pi_g = \left( \frac{\partial \Delta f}{\partial \sigma} + x \frac{\partial \Delta g}{\partial \sigma} \right) (S_+ - S_-) \delta \sigma \]
which means there are still risks, i.e. our portfolio \( f + xg \) is not completely \( \Delta \)-neutral.

Now suppose we have another contingent claim on the same stock to form a new portfolio:
\[ f + xg + yh \]
we can choose \( x, y \), such that

\[ \text{\textit{Vega}-Neutral:} \quad \frac{\partial f}{\partial \sigma} + x \frac{\partial g}{\partial \sigma} + y \frac{\partial h}{\partial \sigma} = 0 \quad \text{i.e.} \quad \mathcal{V}_f + x \mathcal{V}_g + y \mathcal{V}_h = 0 \quad (2a) \]

and \( \Delta \)-Neutral:
\[ \frac{\partial \Delta f}{\partial \sigma} + x \frac{\partial \Delta g}{\partial \sigma} + y \frac{\partial \Delta h}{\partial \sigma} = 0 \quad (2b) \]
then we have hedged away potential mispricing and risks due to misparameterization of \( \sigma \).

**Importance of being nonlinear:** Question: can we use the stock as our third option for hedging, i.e.,
\[ h(S_T) = S_T \]

i.e., the stock itself for our option \( h(S) \)? Note that
\[ \mathcal{V}_h = \frac{\partial S}{\partial \sigma} = 0 \quad \text{a stock has vanishing Vega} \]
and \( \Delta_h = \frac{\partial S_0}{\partial S} = 1 \quad \left[ \text{or } \Delta_h = \frac{S_+ - S_-}{S_+ - S_-} = 1 \right] \]
i.e., a stock has \( \Delta = 1 \)
\[ \implies \text{stock } S \text{ is a linear derivative} \]
Can we use the stock itself for our \( h \)?

Since Eqs. (2) now become
\[ \frac{\partial f}{\partial \sigma} + x \frac{\partial g}{\partial \sigma} + y \cdot 0 = 0 \]
and \[ \frac{\partial \Delta f}{\partial \sigma} + x \frac{\partial \Delta g}{\partial \sigma} + y \cdot 0 = 0 \]
leading to no solution for \( x, \) and \( y \), in general.

The story is just to give you some sense of how issues of hedging arise and how hedging can be done. This simple example illustrates the need for **nonlinear** derivatives for hedging purposes.

**Conclusion:**
Even if a stock price dynamics is not 100% accurate, as long as it is sufficiently close to the true dynamics — meaning both model specification and model parameterization — then we can use a well-balanced (hedged) portfolio to eliminate most of risks.
2.2 Hedging (General Formulation) — Greeks

2.2.1 Greeks

Portfolio value:

\[ \Pi = \Pi (t, S, \sigma, r) \]

where \( t \) is time-to-maturity. Then

\[ \delta \Pi = \frac{\partial \Pi}{\partial t} \delta t + \frac{\partial \Pi}{\partial S} \delta S + \frac{\partial \Pi}{\partial \sigma} \delta \sigma + \frac{\partial \Pi}{\partial r} \delta r \]

\[ + \frac{1}{2} \frac{\partial^2 \Pi}{\partial S^2} (\delta S)^2 + \cdots \]

where \( \delta t \) indicates the changing of time. What are the Greeks? They are no more than

- \( \Theta = \frac{\partial \Pi}{\partial t} \)
- \( \Delta = \frac{\partial \Pi}{\partial S} \)
- \( \nu = \frac{\partial \Pi}{\partial \sigma} \)
- \( \rho = \frac{\partial \Pi}{\partial r} \)
- \( \Gamma = \frac{\partial^2 \Pi}{\partial S^2} \)

Note that a portfolio contains, e.g., stocks, calls, puts, etc. each of which has its own corresponding \( \Delta, \Gamma, \text{etc.} \) For example, for a stock,

\[ \Delta_S = 1 \]
\[ \Gamma_S = 0 \]
\[ \nu_S = 0 \]

2.2.2 Greeks for a European Call/Put:

\[ \Delta_c = \frac{\partial}{\partial S_0} c(S_0, T, K) = N(d_1) \quad \text{(How to evaluate? HW)} \]

\[ \Delta_p = \frac{\partial}{\partial S_0} p(S_0, T, K) = -N(-d_1) = N(d_1) - 1 \]

the second line of which can be seen directly from the put-call parity.

Note that the hedging portfolio in a risk-neutral way would be

\[ \Delta_c S_0 - c \]

With changing \( \Delta_c \), one has to rebalance.
\[ \Delta := \frac{\partial}{\partial S_0} \] — sensitivity to the change of the stock price.

Call: \[ 1 \geq N(d_1) \geq 0 \]  
Put: \[ -1 \leq N(d_1) - 1 \leq 0 \]

\[ \Gamma := \frac{\partial^2}{\partial S_0^2} \] — \( \Delta \)-sensitivity to the change of \( S_0 \)

\[ \Gamma_c = \frac{1}{S_0 \sqrt{2\pi\sigma^2T}} \exp\left[\frac{-d_1^2}{2}\right] > 0 \]  
Put: \[ \Gamma_p = \Gamma_c \]

\( \Gamma_p = \Gamma_c \) can be seen directly from the put-call parity.
\[ \nu := \frac{\partial}{\partial \sigma} \quad \text{sensitivity to the volatility change} \]

Call: \[ V_c = S_0 \sqrt{\frac{T}{2\pi}} \exp \left[ -\frac{d_1^2}{2} \right] > 0, \]

Put: \[ V_p = V_c \]

again, the relation \( V_p = V_c \) can be seen from the put-call parity.

Note that the Black-Scholes formula assumes a constant \( \sigma \). But, if volatility changes, then rebalance is needed.
\( \Theta := \frac{\partial \varepsilon}{\partial t} \) — sensitivity to "time-to-maturity"

Call : \( \Theta_c = -\frac{S_0 \sigma}{2\sqrt{2\pi} T} \exp \left[ -\frac{d_1^2}{2} \right] - r Ke^{-rT} N(d_2) < 0 \)

Put : \( \Theta_p = -\frac{S_0 \sigma}{2\sqrt{2\pi} T} \exp \left[ -\frac{d_1^2}{2} \right] + r Ke^{-rT} N(d_2) = \begin{cases} > 0 \\ < 0 \end{cases} \)

What do we mean by sensitivity to "time-to-maturity" since there is nothing uncertain about time after all? Actually this is another way of looking at Gamma \( \Gamma \).

Recall the Black-Scholes PDE:

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} - rf = 0
\]

i.e.,

\[
\Theta + \frac{1}{2} \sigma^2 S^2 \Gamma + r S \Delta = rf
\]  

(3)

Note that

1. If we need nothing to enter a contract (a portfolio with many options), then

\[
f(0) = 0
\]

If we want to maintain that way, i.e., constant rebalance to ensure \( f = 0 \), then, Eq. (3) yields

\[
\Delta\text{-neutral, } \Theta\text{-neutral } \implies \Gamma\text{-neutral}
\]
2. For a $\Delta$-neutral portfolio, we have
\[ \Theta + \frac{1}{2} \sigma^2 S^2 \Gamma = rf \]
then

large, positive $\Theta \implies \text{large, negative $\Gamma$}

large, negative $\Theta \implies \text{large, positive $\Gamma$}

$\rho := \frac{\partial}{\partial r}$ — sensitivity to the change of interest rate

Call : $\rho_c = TK e^{-rT} N(d_2) > 0$

Put : $\rho_p = -TK e^{-rT} N(-d_2) = TK e^{-rT}[N(d_2) - 1] < 0$

2.2.3 General Hedging:

$\Delta$-Hedge: e.g., $n_1$ units of an option $f_1$ on the same stock.

and the portfolio is
\[ \Pi_1 = n_1 f_1 + n_s S_0 + B \]

where $B$ stands for the amount of bond. $\Delta$-neutral is
\[ \frac{\partial}{\partial S_0} \Pi_1 = 0 \]

i.e.,
\[ n_1 \Delta_1 + n_s = 0 \]

which is insensitive to the change of the stock price. Since,
\[ n_1 f_1 + n_s S_0 + B = V_1 \]
\[ n_1 \Delta_1 + n_s = 0 \]

which are two equations with two unknowns, $n_1, n_s$, there is no more freedom to hedge other Greeks.
**V-Hedge:** However, if we have two options $f_1$, $f_2$ (on the same stock) with $n_1$ and $n_2$ units, respectively, then the value of the portfolio is

$$\Pi_2 = n_1 f_1 + n_2 f_2 + n_s S_0 + B$$

its value is

$$n_1 f_1 + n_2 f_2 + n_s S_0 + B = V_2,$$  \hspace{1cm} (4)

with $\Delta$-neutral position, i.e.,

$$\frac{\partial \Pi_2}{\partial S_0} = 0 \implies n_1 \Delta_1 + n_2 \Delta_2 + n_s = 0$$ \hspace{1cm} (5)

Now we have more freedom for hedging. For example, we can demand $Vega$-neutral, i.e.,

$$\frac{\partial}{\partial \sigma} \Pi_2 = 0 \implies n_1 V_1 + n_2 V_2 = 0$$ \hspace{1cm} (6)

Eqs. (4), (5) and (6) can be solved for the three unknowns, $n_1, n_2, n_3$ — thus, achieving $\Delta$-neutral and $Vega$-neutral, reducing the exposure to the changes or mis-specification of volatility, etc.

Note that

1. By increasing types of options, we can hedge away other kinds of risks described by other Greeks.

2. Dynamic balancing: Often times $\Gamma$ and $Vega$ are monitored but not zeroed out. $\Delta$ is zeroed out daily by rebalancing shares.

3. There is a difficulty of $\Gamma$-neutral and $V$-neutral — which require **nonlinear** derivatives that are traded at competitive prices.

### 2.2.4 Speculation using Greeks

Consider only a European call

$$\delta c = \frac{\partial c}{\partial S} \delta S = \Delta \delta S$$

$$\frac{\delta c}{c} = \frac{\Delta}{c} \delta S = \frac{S \Delta \delta S}{c S}$$
If \[ \frac{S}{c} \Delta \gg 1 \]
then, a percentage change of stock price \( \delta S/S \) can lead to an appreciably large percentage change of option price.

One can also bet on volatility:

\[
\delta \Pi = \frac{\delta \Pi}{\delta \sigma} \delta \sigma = \nu \delta \sigma \\
\frac{\delta \Pi}{\Pi} = \nu \sigma \frac{\delta \sigma}{\sigma}
\]

If a portfolio is constructed in such a way that

\[ \frac{\nu}{\Pi \sigma} \gg 1 \]

then, it is possible to use this portfolio for speculation with high leverage, i.e., a change of volatility is magnified by a factor \( \frac{\nu}{\Pi \sigma} \) in the portfolio price.