6. LIBOR market model

Andrew Lesniewski

March 10, 2008

Contents

1 Introduction

The recently emerging industry standard for interest rates modeling is the LIBOR market model (LMM). Unlike the short rate models discussed in the Lecture 5, it captures the dynamics of the entire curve of interest rates by using dynamic LIBOR forwards as its building blocks. The time evolution of the forwards is given by a set of intuitive stochastic equations in a way which guarantees arbitrage freeness of the process. The model is intrinsically multi-factor, meaning that it captures accurately various aspects of the curve dynamics: parallel shifts, steepenings / flattenings, butterflies, etc.

One of the main difficulties experienced by the pre-LMM term structure models is the fact that they tend to produce unrealistic volatility structures of forward rates. The persistent "hump" occurring in the short end of the volatility curve leads to overvaluation of instruments depending on forward volatility. The LMM offers a solution to this problem by allowing one to impose an approximately stationary volatility and correlation structure of LIBOR forwards. This reflects the view that the volatility structure of interest rates retains its shape over time, without distorting the valuation of instruments sensitive to forward volatility.

On the downside, since the LMM is far less tractable than, for example, the Hull-White model. In addition, it is not Markovian in the sense short rate models are Markovian.

2 LIBOR market model

2.1 Dynamics of the LIBOR market model

We shall consider a sequence of approximately equally spaced dates $0 = T_0 < T_1 < \ldots < T_N$ which will be termed the *standard tenors*. A standard LIBOR forward rate $F_j,\ j = 0, 1, \ldots, N-1$ is associated with a FRA which starts on T_j and matures on T_{j+1}. Usually, it is assumed that $N = 120$ and the F_j's are 3 month LIBOR forward rates. Note that these dates refer to the actual start and end dates of the contracts rather than the LIBOR "fixing dates", i.e. the dates on which the LIBOR rates settle. To simplify the notation, we shall disregard the difference between the contract's start date and the corresponding forward rate's fixing date. Proper implementation, however, must take this distinction into account.

Each LIBOR forward F_j is modeled as a continuous time stochastic process $F_j(t)$. Clearly, this process has the property that it gets killed at $t = T_j$. For future convenience, we define $\gamma : [0,\ T_N] \to \mathbb{Z}$ to be a function given by

$$\gamma(t) = m, \quad \text{if } t \in [T_{m-1},\, T_m)\ .$$

The dynamics of the forward process is driven by an N-dimensional, correlated Wiener process $W_1(t), \ldots, W_N(t)$. We let ρ_{jk} denote the correlation coefficient between $W_j(t)$ and $W_k(t)$:

$$\mathsf{E}\left[dW_j(t)\, dW_k(t)\right] = \rho_{jk} dt\,,$$

where E denotes expected value.

In order to motivate the form of the stochastic differential equations describing the dynamics of the LIBOR forwards, let us first consider the world in which there is no volatility of interest rates. The shape of the forward curve would be set once and for all by a higher authority, and each LIBOR forward would have a constant value $F_j(t) = F_{j0}$. In other words,

$$dF_j(t) = 0,$$

for all j's. The fact that the rates are stochastic forces us to replace this simple dynamical system with a system of stochastic differential equations of the form:

$$dF_j(t) = \Delta_j(F(t), t)\, dt + C_j(F(t), t)\, dW_j(t)\,. \tag{1}$$

As discussed in Lecture 3, the *no arbitrage* requirement of asset pricing forces a relationship between the drift term and the diffusion term: the form of the drift term depends thus on the choice of numeraire.

Recall from Lecture 2 that F_k is a martingale under the T_k-forward measure Q_k, and so its dynamics reads:

$$dF_k(t) = C_k(F_k(t), t)\, dW_k(t)\,,$$

where $C_k(F_k(t), t)$ is an instantaneous volatility function which will be defined later. For $j \neq k$,

$$dF_j(t) = \Delta_j(F(t), t)\, dt + C^j(F_j(t), t)\, dW_j(t)\,.$$

Since the j-th LIBOR forward settles at T_{j-1}, the process for $F_j(t)$ is killed at $t = T_{j-1}$. We shall determine the drifts $\Delta_j(F(t), t)$ by requiring lack of arbitrage.

Let us first assume that $j < k$. The numeraires for the measures Q_j and Q_k are the prices $P(t, T_j)$ and $P(t, T_k)$ of the zero coupon bonds expiring at T_j and T_k, respectively. Explicitly,

$$P(t, T_j) = \prod\nolimits_{\gamma(t) \leq i \leq j} \frac{1}{1 + \delta_i F^i(t)}\,. \tag{2}$$

Since the drift of $F^j(t)$ under Q_j is zero, formula (30) of Lecture 3 yields:

$$\begin{aligned}
\Delta_j(F(t), t) &= -\left\{F_j, \frac{P(t, T_j)}{P(t, T_k)}\right\}(t) \\
&= -\left\{F_j, \prod\nolimits_{j+1 \leq i \leq k} (1 + \delta_i F_i)\right\}(t) \\
&= -\frac{d}{dt}\left\langle F_j, \log \prod\nolimits_{j+1 \leq i \leq k} (1 + \delta_i F_i)\right\rangle(t) \\
&= -C_j(F_j(t), t) \sum\nolimits_{j+1 \leq i \leq k} \frac{\rho_{ji}\delta_i C_i(F_i(t), t)}{1 + \delta_i F_i(t)}\,.
\end{aligned}$$

Similarly, for $j > k$, we find that

$$\Delta_j\left(F\left(t\right),t\right) = C^j\left(F_j\left(t\right),t\right)\sum\nolimits_{k+1\leq i\leq j}\frac{\rho_{ji}\delta_i C_i\left(F_i\left(t\right),t\right)}{1+\delta_i F_i\left(t\right)}\ .$$

We can thus summarize the above discussion as follows. In order to streamline the notation, we let $dW\left(t\right) = dW^{\mathsf{Q}_k}\left(t\right)$ denote the Wiener process under the measure Q_k. Then the dynamics of the LMM is given by the following system of stochastic differential equations:

$$\begin{aligned} dF^j\left(t\right) &= C^j\left(F^j\left(t\right),t\right) \\ &\times\begin{cases} -\sum\nolimits_{j+1\leq i\leq k}\dfrac{\rho_{ji}\delta_i C_i\left(F^i\left(t\right),t\right)}{1+\delta_i F^i\left(t\right)}\,dt + dW_j\left(t\right), & \text{if } j<k, \\ dW_j\left(t\right), & \text{if } j=k, \\ \sum\nolimits_{k+1\leq i\leq j}\dfrac{\rho_{ji}\delta_i C_i\left(F^i\left(t\right),t\right)}{1+\delta_i F_i\left(t\right)}\,dt + dW_j\left(t\right), & \text{if } j>k\,. \end{cases} \end{aligned} \tag{3}$$

These equations have to be supplied with initial values for the LIBOR forwards:

$$F_j\left(0\right) = F_0^j, \tag{4}$$

where F_0^j is the current value of the forward which is implied by the current yield curve.

In addition to the forward measures discussed above, it is convenient to use the spot measure. It is expressed in terms of the numeraire:

$$P\left(t\right) = \frac{P\left(t,T_{\gamma(t)-1}\right)}{\prod_{1\leq i\leq\gamma(t)-1}P\left(T_{i-1},T_i\right)}\ . \tag{5}$$

Under the spot measure, the LMM dynamics reads:

$$dF_j\left(t\right) = C_j\left(F_j\left(t\right),t\right)\left(\sum_{\gamma(t)\leq i\leq j}\frac{\rho_{ji}\delta_i C_i\left(F_i\left(t\right),t\right)}{1+\delta_i F_i\left(t\right)}\,dt + dW_j\left(t\right)\right). \tag{6}$$

2.2 Factor reduction

In a market where the forward curve spans 30 years, there are 120 quarterly LIBOR forwards and thus 120 stochastic factors. So far we have not imposed any restrictions on the number of these factors, and thus the number of Brownian motions driving the LIBOR forward dynamics is equal to the number of forwards. Having a large number of factors poses severe problems with the model's implementation.

On the numerical side, the "curse of dimensionality" kicks in, leading to unacceptably slow performance. On the financial side, the parameters of the model are severely underdetermined and the calibration of the model becomes unstable.

We are thus led to the assumption that only a small number d of independent Brownian motions $Z_a(t)$, $a = 1, \ldots, d$, with

$$\mathsf{E}\left[dZ_a(t)\, dZ_b(t)\right] = \delta_{ab} dt\,, \tag{7}$$

should drive the process. Typically, $d = 1, 2, 3$, or 4. We set

$$dW_j(t) = \sum_{1 \le a \le d} U_{ja}\, dZ_a(t)\,, \tag{8}$$

where U is an $N \times d$ matrix with the property that UU' is close to the correlation matrix. Of course, it is in general impossible to have $UU' = \rho$. We can easily rewrite the dynamics of the model in terms of the independent Brownian motions:

$$dF_j(t) = \Delta_j(F(t), t)\, dt + \sum_{1 \le a \le d} B_{ja}(F_j(t),\, t)\, dZ_a(t)\,, \tag{9}$$

where

$$B_{ja}(F_j(t),\, t) = U_{ja}\, C_j(F_j(t),\, t)\,. \tag{10}$$

We shall call this system the *factor reduced* LMM dynamics.

2.3 Low noise solution

There is no known closed form solution to the initial value problem (3) - (4), not even in the case of constant C_j. In this section, we shall present an approximation which will be frequently used throughout the manuscript.

This approximation is obtained by means of a low noise expansion. In order to generate the expansion we introduce a formal "smallness parameter" ε in front of the diffusion coefficients $B_{ja}(F, t)$ This parameter is set to 1 at the end of the computation. Since the drift coefficients are quadratic in the diffusion coefficients, we have to multiply $\Delta_j(F, t)$ by ε^2. Straightforward calculations yield then the following (asymptotic) expansion:

$$\begin{aligned} F_j(t) = F_{0j} &+ \sum_{1 \le a \le d} \int_0^t B_{ja}(F_0,\, s)\; dZ_a(s) + \int_0^t \Delta_j(F_0,\, s)\, ds \\ &+ \sum_{\substack{1 \le a,b \le d \\ 1 \le k \le N}} \iint_{0 \le u \le s \le t} B_{ka}(F_0,\, u)\, \frac{\partial B_{jb}(F_0,\, s)}{\partial F_k}\, dZ_a(u)\, dZ_b(s) \\ &+ \ldots\,. \end{aligned} \tag{11}$$

Now, let E denote expected value with respect to one of the martingale measures defined in Section 2.1. Expansion (11) implies the following asymptotic expansion for the expected value of $\mathsf{E}\left[F_j\left(t\right)\right]$:

$$\begin{aligned}\mathsf{E}\left[F_j\left(t\right)\right] = F_{0j} + \int_0^t \Delta_j\left(F_0,\, s\right) ds \\ + \sum_{\substack{1\leq a,b\leq d \\ 1\leq k\leq N}} \rho_{ab} \int_0^t B_{ka}\left(F_0,\, s\right) \frac{\partial B_{jb}\left(F_0,\, s\right)}{\partial F_k}\, ds + \dots\,.\end{aligned} \tag{12}$$

This formula shows that the naive expected value, namely the current forward, is adjusted by a convexity correction which depends on the market volatility. We also have the following useful asymptotic expansion of the covariance $\mathsf{Cov}\left[F_j\left(t\right), F_k\left(t\right)\right]$:

$$\begin{aligned}\mathsf{Cov}\left[F_j\left(t\right), F_k\left(t\right)\right] = \sum_{1\leq a,b\leq d} \rho_{ab} \int_0^t B_{ja}\left(F_0,\, s\right) B_{kb}\left(F_0,\, s\right)\, ds \\ + \sum_{\substack{1\leq a,b,a',b'\leq d \\ 1\leq i,i'\leq N}} \rho_{aa'}\rho_{bb'} \int_0^t \int_0^t B_{ia}\left(F_0,\, s\right) B_{i'a'}\left(F_0,\, s'\right) \\ \times \frac{\partial B_{jb}\left(F_0,\, s\right)}{\partial F_i} \frac{\partial B_{kb'}\left(F_0,\, s'\right)}{\partial F_{i'}}\, ds\, ds' + \dots\,.\end{aligned} \tag{13}$$

Again, the first term in the formula above is what one would naively expect. The second term, as well as the suppressed higher order terms, constitute a convexity correction to this quantity.

3 Calibration of the LMM

Calibration (to a selected collection of benchmark instruments) is a choice of the model parameters so that the model reprices the benchmark instruments to a desired accuracy. The choice of the calibrating instruments is dictated by the characteristics of the portfolio to be managed by the model.

An important feature of the LMM is that it leads to pricing formulas for caps and floors which are consistent with the market practice of quoting the prices of these products in terms of Black's model. This makes the calibration of the LMM to caps and floors very easy. On the other hand, from the point of view of the LMM, swaptions are exotic structures whose fast pricing poses serious challenges. In this section we describe our strategy of dealing with these issues.

3.1 Approximate valuation of swaptions

A key ingredient of any efficient calibration methodology for the LMM is rapid and accurate swaption valuation. A swap rate is a non-linear function of the underlying LIBOR forward rates. The stochastic differential equation for the swap rate implied by the LMM cannot be solved in closed form, and thus pricing swaptions within the LMM requires Monte Carlo simulations. This poses a serious issue for efficient model calibration, as such simulations are very time consuming.

Let us describe an approximation which can be used to calibrate the model. Our method uses the low noise expansion of Section 2.3. We consider a standard forward starting swap. The start date of the swap is denoted by T_m, and its end date is denoted by T_n. Recall that the level function of the swap is defined by:

$$L^{mn}(t) = \sum_{m \leq j \leq n-1} \alpha_j P(t, T_{j+1}), \tag{14}$$

where α_j are the day count fractions for fixed rate payments, and where $P(t, T_j)$ is the time t value of \$1 paid at time T_j. Typically, the payment frequency on the fixed leg is not the same as that on the floating leg[1] (which we continue to denote by δ_j). This fact causes a bit of a notational nuisance but needs to be taken properly into account for accurate pricing. The forward swap rate is given by:

$$\begin{aligned} S^{mn}(t) &= \frac{P(t, T_m) - P(t, T_n)}{L^{mn}(t)} \\ &= \frac{1}{L^{mn}(t)} \sum_{m \leq j \leq n-1} \delta_j F_j(t) P(t, T_{j+1}) \ . \end{aligned} \tag{15}$$

In order to lighten up the notation, we will suppress the superscripts mn throughout the remainder of this chapter.

A straightforward calculation shows that the dynamics of the swap rate process in the LIBOR market model can be written in the form:

$$dS(t) = \Omega(F, t)\, dt + \sum_{m \leq j \leq n-1} \Lambda_j(F, t)\, dW_j(t), \tag{16}$$

where

$$\Omega = \sum_{m \leq j \leq n-1} \frac{\partial S}{\partial F_j} \Delta_j + \frac{1}{2} \sum_{m \leq j,k \leq n-1} \rho_{jk} \frac{\partial^2 S}{\partial F_j \partial F_k} C_j C_k \, , \tag{17}$$

[1] Remember, the default convention on US dollar swaps is a semiannual $30/360$ fixed leg versus a quarterly act$/360$ floating leg.

and

$$\Lambda_j = \frac{\partial S}{\partial F_j} C_j. \tag{18}$$

In order to be able to use this dynamics effectively, we have to approximate it by quantities with tractable analytic forms.

The simplest approximation consists in replacing the values of the stochastic forwards $F_j(t)$ by their initial values F_{j0}. This amounts to "freezing" the curve at its current shape. Within this approximation, the coefficients in the diffusion process (16) for the swap rate are deterministic:

$$\Lambda_j(F, t) \approx \Lambda_j(F_0, t), \tag{19}$$

and

$$\Omega(F, t) \approx \Omega(F_0, t). \tag{20}$$

The stochastic differential equation can be solved in closed form,

$$S(t) = S_0 + \int_0^t \Omega(F_0, s)\, ds + \sum_{m \le j \le n-1} \int_0^t \Lambda_j(F_0, s)\, dW_j(s). \tag{21}$$

This implies that, within the frozen curve approximation, the expected value of the swap rate is given by

$$\mathsf{E}[S(t)] = S_0 + \int_0^t \Omega(F_0, s)\, ds, \tag{22}$$

and its variance is

$$\mathsf{Var}[S(t)] = \sum_{m \le j, j' \le n-1} \rho_{jj'} \int_0^t \Lambda_j(F_0, s)\, \Lambda_{j'}(F_0, s)\, ds. \tag{23}$$

Now, the normal volatility ζ_{mn} of a swaption expiring T_m years from now is

$$\zeta_{mn} = \sqrt{\frac{1}{T_m} \mathsf{Var}[S^{mn}]}\,. \tag{24}$$

Consequently, its frozen curve approximation $\zeta_{0,mn}$ is given by

$$\zeta_{0,mn}^2 = \frac{1}{T_m} \sum_{m \le j, j' \le n-1} \rho_{jj'} \int_0^t \Lambda_j(F_0, s)\, \Lambda_{j'}(F_0, s)\, ds\,. \tag{25}$$

This formula is easy to implement in code, and leads to reasonably accurate results.

We can develop a small noise expansion of the swap rate in order to go beyond the frozen curve approximation. Substituting (11) into (16) yields

$$\begin{aligned} dS(t) &= \Omega(F_0,t)\,dt + \sum_{m\le j\le n-1} \Lambda_j(F_0,t)\,dW_j(t) \\ &\quad + \sum_{m\le j,k\le n-1} \frac{\partial\Lambda_j}{\partial F_k}(F_0,t)\left(\int_0^t C_k(F_0,u)\,dW_k(u)\right) dW_j(t)\,, \end{aligned} \tag{26}$$

and thus

$$\begin{aligned} S(t) &= S_0 + \int_0^t \Omega(F_0,s)\,ds + \sum_{m\le j\le n-1} \int_0^t \Lambda_j(F_0,s)\,dW_j(s) \\ &\quad + \sum_{m\le j,k\le n-1} \int_0^t \frac{\partial\Lambda_j}{\partial F_k}(F_0,s) \int_0^s C_k(F_0,u)\,dW_k(u)\,dW_j(s) \\ &\quad + \dots\,. \end{aligned} \tag{27}$$

This implies that the expected value of the swap rate is

$$\mathsf{E}[S(t)] = S_0 + \int_0^t \Omega(F_0,s)\,ds + \dots, \tag{28}$$

and its variance is given by

$$\begin{aligned} \mathsf{Var}[S(t)] &= \sum_{m\le j,j'\le n-1} \rho_{jj'} \int_0^t \Lambda_j(F_0,s)\,\Lambda_{j'}(F_0,s)\,ds \\ &\quad + \sum_{\substack{m\le j,k\le n-1 \\ m\le j',k'\le n-1}} \rho_{jj'}\rho_{kk'} \int_0^t\int_0^t \frac{\partial\Lambda_j}{\partial F_k}(F_0,s)\,\frac{\partial\Lambda_{j'}}{\partial F_{k'}}(F_0,s) \\ &\qquad \times C_k(F_0,u)\,C_{k'}(F_0,u)\;ds\,du + \dots\,. \end{aligned} \tag{29}$$

Reinstating the superscript on the swap rate, we shall rewrite this expansion as

$$\mathrm{Var}[S^{mn}] = \zeta_{0,mn}^2\,T_m + \eta_{0,mn}\,T_m^2 + \dots\;, \tag{30}$$

where $\zeta_{0,mn}$ is defined in (25), and where $\eta_{0,mn}$ is given by

$$\begin{aligned} \eta_{0,mn} &= \frac{1}{T_m^2} \sum_{\substack{m\le j,k\le n-1 \\ m\le j',k'\le n-1}} \rho_{jj'}\rho_{kk'} \int_0^{T_m}\int_0^{T_m} \frac{\partial\Lambda_j}{\partial F_k}(F_0,s)\,\frac{\partial\Lambda_{j'}}{\partial F_{k'}}(F_0,s) \\ &\qquad \times C_k(F_0,u)\,C_{k'}(F_0,u)\;ds\,du\,. \end{aligned} \tag{31}$$

Consequently, the next order swaption normal volatility, which we denote by $\zeta_{1,mn}$ is given by

$$\begin{aligned}\zeta_{1,mn} &= \sqrt{\zeta_{0,mn}^2 + \eta_{0,mn}\, T_m} \\ &= \zeta_{0,mn} + \frac{1}{2}\,\eta_{0,mn}\, T_m + \dots \,.\end{aligned} \tag{32}$$

For all practical purposes, the formula above gives a sufficient approximation to the swaption volatility. It also is a good compromise between accuracy, ease of implementation in computer code, and computational performance.

3.2 Structure of instantaneous volatility

So far we have been working with a general instantaneous volatility $C_j\left(F_j\left(t\right), t\right)$ for the forward $F_j\left(t\right)$. In the implementation, we assume $C_j\left(F_j\left(t\right), t\right)$ to be one of the following standard models:

$$C^j\left(F_j\left(t\right), t\right) = \begin{cases} \sigma_j\left(t\right) & \text{(normal model),} \\ \sigma_j\left(t\right) F_j\left(t\right)^{\beta_j} & \text{(CEV model),} \\ \sigma_j\left(t\right) F_j\left(t\right) & \text{(lognormal model),} \\ \sigma_j\left(t\right) F_j\left(t\right) + \delta_j & \text{(shifted lognormal model),} \end{cases} \tag{33}$$

where the functions $\sigma_j\left(t\right)$ are deterministic, and where $0 \leq \beta_j \leq 1$, $\delta_j \geq 0$.

For the purpose of calibration we require that the deterministic volatility components $\sigma_j\left(t\right)$ are piecewise constant. That leads to the following parametrization of the instantaneous volatility:

$\sigma^j\left(t\right) \diagdown t \in$	$[T_0, T_1)$	$[T_1, T_2)$	$[T_2, T_3)$	$\ldots$	$[T_{N-1}, T_N)$
$\sigma^0\left(t\right)$	0	0	0	$\ldots$	0
$\sigma^1\left(t\right)$	$\sigma_{1,0}$	0	0	$\ldots$	0
$\sigma^2\left(t\right)$	$\sigma_{2,0}$	$\sigma_{2,1}$	0	$\ldots$	0
$\sigma^3\left(t\right)$	$\sigma_{3,0}$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sigma^{N-1}\left(t\right)$	$\sigma_{N-1,0}$	$\sigma_{N-1,1}$	$\sigma_{N-1,2}$	$\ldots$	0

The table above contains 7140 parameters (assuming $N = 120$), and the problem is vastly overparametrized.

A natural remedy to the overparameterization problem is assuming that the instantaneous volatility is *stationary*, i.e.,

$$\begin{aligned}\sigma_{j,i} &= \sigma_{j-i,0} \\ &\equiv \sigma_{j-i}\,,\end{aligned} \tag{34}$$

for all $i < j$. This assumption appears natural and intuitive, as it implies that the structure of cap volatility will look in the future exactly the same way as it does currently. Consequently, the "forward volatility problem" plaguing the traditional terms structure models would disappear. With the stationary volatility assumption we have the following parametrization of the instantaneous volatility structure:

$\sigma^j(t) \diagdown t \in$	$[T_0, T_1)$	$[T_1, T_2)$	$[T_2, T_3)$	$\dots$	$[T_{N-1}, T_N)$
$\sigma^0(t)$	0	0	0	$\dots$	0
$\sigma^1(t)$	σ_1	0	0	$\dots$	0
$\sigma^2(t)$	σ_2	σ_1	0	$\dots$	0
$\sigma^3(t)$	σ_3	σ_2	σ_1	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sigma^{N-1}(t)$	σ_{N-1}	σ_{N-2}	σ_{N-3}	$\dots$	0

Despite its appeal, this assumption is not suitable for accurate calibration of the model. The financial reason behind this fact appears to be the phenomenon of *mean reversion of long term rates*. Unlike in the Vasicek style models, it is impossible to take this phenomenon into account by adding an Ornstein - Uhlenbeck style drift term to the LMM dynamics as this would violate the arbitrage freeness of the model. On the other hand, one can achieve a similar effect by suitably modifying the instantaneous volatility function.

Specifically, we introduce a *volatility kernel* function $K(\tau, \lambda)$ whose role is to account for the deviation of the instantaneous volatility form the purely stationary model. A particularly convenient form of the volatility kernel is

$$K(\tau, \lambda) = \exp(-\lambda\tau). \tag{35}$$

For each maturity T_j we choose a parameter λ_j, and set

$$K_{j,i} = K(T_j - T_i, \lambda_j), \tag{36}$$

and we assume the following structure of the instantaneous volatility:

$\sigma^j(t) \diagdown t \in$	$[T_0, T_1)$	$[T_1, T_2)$	$[T_2, T_3)$	$\dots$	$[T_{N-1}, T_N)$
$\sigma^0(t)$	0	0	0	$\dots$	0
$\sigma^1(t)$	$\sigma_1 K_{1,0}$	0	0	$\dots$	0
$\sigma^2(t)$	$\sigma_2 K_{2,0}$	$\sigma_1 K_{2,1}$	0	$\dots$	0
$\sigma^3(t)$	$\sigma_3 K_{3,0}$	$\sigma_2 K_{3,1}$	$\sigma_1 K_{3,2}$	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sigma^{N-1}(t)$	$\sigma_{N-1} K_{N-1,0}$	$\sigma_{N-2} K_{N-1,1}$	$\sigma_{N-3} K_{N-1,2}$	$\dots$	0

The lower triangular matrix above, LMM's internal representation of volatility, is referred to as the *LMM volatility surface*.

3.3 The structure of the correlation matrix

The central issue is to calibrate the model, at the same time, to the cap / floor and swaption markets in a stable and consistent manner. An important part of this process is determining the correlation matrix $\rho = \{\rho_{jk}\}_{0\leq j,k\leq N-1}$. The dimensionality of ρ is $N\,(N+1)\,/2$, clearly far to high to assure a stable calibration procedure.

A convenient approach to correlation modeling is to use a parameterized form of ρ_{ij}. An intuitive and flexible parametrization is given by the formula:

$$\rho_{ij} = \rho_\infty + (1-\rho_\infty)\exp\left(-\frac{\lambda\,|T_i - T_j|}{1+\kappa\min\left(T_i, T_j\right)}\right), \tag{37}$$

where ρ_∞ is the asymptotic level of correlations, λ is a the decay rate of correlations, and κ is an asymmetry parameter[2]. The parameters in this formula can be calibrated by using, for example, historical data.

3.4 Optimization

In order to calibrate the model we seek instantaneous volatility parameters σ_i so that to fit the at the money caplet and swaption volatilities. These can be expressed in terms of the instantaneous volatilities are as follows. The at the money volatility of the caplet expiring at T_m is given by:

$$\begin{aligned}\zeta_m\left(\sigma_1,\ldots,\sigma_m,\lambda_m\right)^2 &= \frac{1}{T_m}\sum_{0\leq i\leq m-1}\sigma_{m-i}^2\int_{T_i}^{T_{i+1}}K\left(T_m-t,\lambda_m\right)^2 dt\\ &\approx \frac{1}{T_m}\sum_{0\leq i\leq m-1}\sigma_{m-i}^2 K_{m,i}^2\left(T_{i+1}-T_i\right).\end{aligned} \tag{38}$$

The at the money volatility of the swaption expiring at T_m into a swap maturing at T_n is approximately equal to

$$\begin{aligned}\zeta_{m,n}\left(\sigma,\lambda_m,\ldots,\lambda_{n-1}\right)^2 = \frac{1}{T_m}&\sum_{m\leq j,k\leq n-1}\rho_{jk}\Lambda_{j;m,n}\Lambda_{k;m,n}\\ &\times\sum_{i=0}^{m-1}\sigma_{j-i}\sigma_{k-i}K_{j,i}K_{k,i}\left(T_{i+1}-T_i\right).\end{aligned} \tag{39}$$

[2]A word of caution is in order: this parametrization produces a matrix that is only approximately positive definite.

Here, Λ_j are simply rescaled versions of the corresponding functions which were defined and calculated asymptotically in Section 3.1.

The objective function is given by:

$$\begin{aligned}\mathcal{L}(\sigma, \lambda) = &\frac{1}{2}\sum_m w_m \left(\zeta_m(\sigma, \lambda) - \overline{\zeta}_m\right)^2 \\ &+ \frac{1}{2}\sum_{m,n} w_{m,n}\left(\zeta_{m,n}(\sigma, \lambda) - \overline{\zeta}_{m,n}\right)^2 \\ &+ \frac{1}{2}\,\alpha\sum_j (\Delta\sigma)_j^2\,,\end{aligned} \tag{40}$$

where $\overline{\zeta}_m$ and $\overline{\zeta}_{m,n}$ are the market observed caplet and swaption volatilities. The coefficients w_m and $w_{m,n}$ are weights which allow the user select the degree of accuracy of calibration of each of the instruments. Finally, the last term is a Tikhonov style regularization added in order to maintain stability of the calibration: it is proportional to the square of magnitude of the discretized second derivative of the σ_j's.

4 Generating Monte Carlo paths for the LMM

The LMM does not allow for a natural implementation based on recombining trees, and thus all valuations have to be performed via Monte Carlo simulations. We shall describe two numerical schemes for generating Monte Carlo paths for the LMM: Euler's scheme and Milstein's scheme. They both consist in replacing the infinitesimal differentials by suitable finite differences.

We choose a sequence of event dates $t_0, t_1, \ldots, t_m$, and denote by $F_{jn} \simeq F_j(t_n)$ the approximate solution. We also set

$$\begin{aligned}\Delta_{jn} &= \Delta_j(F_n,\, t_n)\,,\\ B_{jan} &= B_{ja}(F_{jn},\, t_n)\,,\end{aligned} \tag{41}$$

and $\delta t_n = t_{n+1} - t_n$. The two discretization schemes read:

(a) *Euler's scheme* can be written down immediately:

$$F_{j,n+1} = F_{jn} + \Delta_{jn}\delta t_n + \sum_{1\leq a\leq d} B_{jan}\,\delta Z_{na}\,, \tag{42}$$

where $\delta Z_{na} = Z_a(t_{n+1}) - Z_a(t_n)$ is the discretized Brownian motion, see Appendix. Euler's scheme is of order of convergence $1/2$ meaning that the approximate solution converges in a suitable norm to the actual solution at the rate of $\delta t^{1/2}$, as $\delta t \equiv \max\, \delta t_n \to 0$.

(b) *Milstein's scheme* is a refinement of Euler's scheme. In order to lighten up the notation, let us define:

$$\Upsilon_{jabn} \equiv B_{ja}\left(F_{jn}, t_n\right) \frac{\partial B_{jb}\left(F_{jn}, t_n\right)}{\partial F_j} . \tag{43}$$

Then Milstein's scheme for the LMM reads:

$$\begin{aligned} F_{j,n+1} = F_{jn} &+ \left(\Delta_{jn} - \frac{1}{2} \sum_{1 \leq a \leq d} \Upsilon_{jaan}\right) \delta t_n \\ &+ \sum_{1 \leq a \leq d} B_{jan}\, \delta Z_{na} + \frac{1}{2} \sum_{1 \leq a,b \leq d} \Upsilon_{jabn}\, \delta Z_{na}\, \delta Z_{nb} . \end{aligned} \tag{44}$$

Milstein's scheme is of order of convergence 1 meaning that the approximate solution converges in a suitable norm to the actual solution at the rate of δt, as $\delta t \to 0$.

5 Valuation and risk management with LMM

Time permits us to make very general remarks about the specifics of the uses of the LIBOR market model in financial practice.

5.1 Valuation

Valuations within the LMM are based on the fundamental theorem of asset pricing theory. Let $\mathfrak{N}(t)$ be a numeraire, and let Q denote the martingale measure associated with $N(t)$. The fundamental pricing theorem (see Lecture 3) states that the time t price of an asset $V(t)$ is given by:

$$V(t) = \mathfrak{N}(t)\, \mathsf{E}^{\mathsf{Q}}\left[\frac{V(T)}{\mathfrak{N}(T)} \,\Big|\, \mathcal{F}_t\right] . \tag{45}$$

For our purposes, Q is either one of the forward measures or the spot measure. The conditional expected value in (45) is calculated by means of Monte Carlo simulations.

5.2 Managing delta risk

This is the most important risk factor (and, arguably, the easiest to hedge). The most natural methodology for calculating the portfolio sensitivity in the context of the LMM is the ridge regression method explained in Section 4.2 of Lecture 1. This method also yields the hedge ratios in terms of the hedging portfolio of vanilla instruments.

5.3 Managing vega risk

In order to quantify the vega risk we have to first design appropriate volatility scenarios. As explained in Section 3.2, LMM has its internal representation $\mathfrak{S}$ of the volatility surface. We construct volatility micro scenarios by accessing $\mathfrak{S}$ and shifting selected non-overlapping segments. Let us call these scenarios

$$\mathfrak{S}_0, \mathfrak{S}_1, \ldots, \mathfrak{S}_q, \tag{46}$$

with $\mathfrak{S}_0 = \mathfrak{S}$ being the base scenario. Next, we choose a hedging portfolio Π_{hedge} which may consist of liquid instruments such as swaptions, caps and floors, Eurodollar options, or other instruments).

The rest is a *verbatim* repeat of the delta story. We calculate the sensitivities of the portfolio to the volatility scenarios (46). We calculate the sensitivities of the hedging portfolio to the volatility scenarios. Finally, we use ridge regression to find the hedge ratios. This method of managing the vega risk works remarkably well and allows one, in particular, to separate the exposure to swaptions from the exposure to caps / floors.

A Generating the Wiener process

There exist many more of less refined methods for simulating a Wiener process; here we describe two of them.

The *random walk method* is easy to implement at the expense of being rather noisy. It represents a Wiener process as a random walk sampled at a finite set of event dates $t_0 < t_1 < \ldots < t_m$:

$$\begin{aligned} W(t_{-1}) &= 0, \\ W(t_n) &= W(t_{n-1}) + \sqrt{t_n - t_{n-1}}\, \xi_n, \quad n = 0, \ldots, m, \end{aligned} \tag{47}$$

where $t_{-1} = 0$, and where ξ_n are i.i.d. random variables with $\xi_n \sim N(0, 1)$. A good method of generating the ξ_n's is to first generate a sequence of uniform pseudorandom numbers u_n (using, say, the Mersenne twister algorithm), and then set

$$\xi_n = N^{-1}(u_n), \tag{48}$$

where $N^{-1}(x)$ is the inverse cumulative normal function.

The *spectral decomposition method* generally leads to much better performance than the random walk method. It assures that the simulated process has the same covariance matrix as the Wiener process $W(t)$ sampled at $t_0, t_1, \ldots, t_m$.

The latter is explicitly given by:

$$\begin{aligned} \mathcal{C}_{ij} &= \mathsf{E}\left[W(t_i)\, W(t_j)\right] \\ &= \min(t_i,\, t_j)\,. \end{aligned} \tag{49}$$

Consider the eigenvalue problem for $\mathcal{C}$:

$$\mathcal{C}E_j = \lambda_j E_j, \qquad j = 0,\, \ldots,\, m, \tag{50}$$

with orthonormal E_j's. Since the covariance matrix $\mathcal{C}$ is positive definite, all of its eigenvalues λ_j are non-negative, and we will assume that

$$\lambda_0 \geq \ \ldots \ \geq \lambda_m \geq 0. \tag{51}$$

We will denote the n-th component of the vector E_j by $E_j(t_n)$, and consider the random variable

$$W(t_n) = \sum_{0 \leq j \leq m} \sqrt{\lambda_j}\, E_j(t_n)\, \xi_j, \tag{52}$$

where ξ_j are, again, i.i.d. random variables with $\xi_j \sim N(0,\, 1)$. These numbers are best calculated by applying the inverse cumulative normal function to a sequence of Sobol numbers. Alternatively, one could use a sequence of uniform pseudorandom numbers; this, however, leads to a significantly higher sampling variance. Then, for each $n = 0,\, \ldots,\, m$, $W(t_n) \sim N(0,\, t_n)$, and

$$\begin{aligned} \mathsf{E}\left[W(t_i)\, W(t_j)\right] &= \sum_{0 \leq k \leq 0} \lambda_k\, E_k(t_i)\, E_k(t_j,) \\ &= \mathcal{C}_{ij}. \end{aligned} \tag{53}$$

We can thus regard $W(t_n)$ a realization of the discretized Wiener process[3]. In practice, we may want to use only a certain portion of the spectral representation (52) by truncating it at some $p < m$. This eliminates the *high frequencies* from $W(t_n)$, and lowers the sampling variance. The price for this may be systematically lower accuracy.

References

[1] Brigo, D., and Mercurio, F.: *Interest Rate Models - Theory and Practice*, Springer Verlag (2006).

[2] Hull, J.: *Options, Futures and Other Derivatives* Prentice Hall (2005).

[3] This realization of the discretized Wiener process is related to the well known Karhounen-Loeve expansion of the (continuous time) Wiener process.