4. Convexity

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1 Convexity corrections

In finance, *convexity* is a broadly understood and non-specific term for nonlinear behavior of the price of an instrument as a function of evolving markets. Oftentimes, financial convexities are associated with some sort of optionality embedded in the instrument. In this lecture we will focus on a small class of convexities which arise in interest rates modeling.

Such convex behaviors manifest themselves as *convexity corrections* to various popular interest rates and they can be blessings and nightmares of market practitioners. From the perspective of financial modeling they arise as the results of valuation done under the "wrong" martingale measure.

Throughout this lecture we will be making careful notational distinction between stochastic processes, such as prices of zero coupon bonds, and their current (known) values. The latter will be indicated by the subscript 0. Thus $P_0(t,T)$ denotes the current value of the forward discount factor, while $P(t,T)$ denotes the time t value of the stochastic process describing the price of the zero coupon bond maturing at T.

2 LIBOR in arrears

Imagine a swap on which LIBOR pays on the start of the accrual period T, rather than at its end date T_{mat}. The PV of such a LIBOR payment is then

$$\mathsf{PV} = P_0(0,T)\,\mathsf{E}^{\mathsf{Q}_T}\left[F(T,T_{\text{mat}})\right], \tag{1}$$

where, as usual, Q_T denotes the T-forward measure. The expected value is clearly taken with respect to the wrong martingale measure! The "natural" measure is the T_{mat}-forward measure. Applying Girsanov's theorem,

$$P_0(0,T)\,\mathsf{E}^{\mathsf{Q}_T}\left[F(T,T_{\text{mat}})\right] = P_0(0,T_{\text{mat}})\,\mathsf{E}^{\mathsf{Q}_{T_{\text{mat}}}}\left[\frac{F(T,T_{\text{mat}})}{P(T,T_{\text{mat}})}\right],$$

and thus the LIBOR in arrears forward is given by:

$$\begin{aligned}
&\mathsf{E}^{\mathsf{Q}_T}\left[F(T,T_{\text{mat}})\right]\\
&\quad = \mathsf{E}^{\mathsf{Q}_{T_{\text{mat}}}}\left[F(T,T_{\text{mat}})\,\frac{P_0(T,T_{\text{mat}})}{P(T,T_{\text{mat}})}\right]\\
&\quad = \mathsf{E}^{\mathsf{Q}_{T_{\text{mat}}}}\left[F(T,T_{\text{mat}})\right] + \mathsf{E}^{\mathsf{Q}_{T_{\text{mat}}}}\left[F(T,T_{\text{mat}})\left(\frac{P_0(T,T_{\text{mat}})}{P(T,T_{\text{mat}})} - 1\right)\right].
\end{aligned}$$

The first term on the right hand side is simply the LIBOR forward, while the second term is the in arrears convexity correction, which we shall denote by $\Delta(T,T_{\text{mat}})$, i.e.,

$$\mathsf{E}^{\mathsf{Q}_T}\left[F(T,T_{\text{mat}})\right] = F_0(T,T_{\text{mat}}) + \Delta(T,T_{\text{mat}}).$$

Let us evaluate this correction using Black's model, i.e.

$$F(T,T_{\text{mat}}) = F_0(T,T_{\text{mat}})\,e^{\sigma W(T) - \frac{1}{2}\sigma^2 T}.$$

Key to the calculation will be the fact that

$$\mathsf{E}\left[e^{aW(t)}\right] = e^{\frac{1}{2}a^2 t} \tag{2}$$

We have

$$P(T,T_{\text{mat}}) = \frac{1}{1+\delta F(T,T_{\text{mat}})},$$

where δ is the coverage factor for the period $[T,\ T_{\mathrm{mat}}]$, and thus, using (2),

$$\mathsf{E}^{Q_{T_{\mathrm{mat}}}}\left[\frac{F\left(T,T_{\mathrm{mat}}\right)}{P\left(T,T_{\mathrm{mat}}\right)}\right]=F_0\left(T,T_{\mathrm{mat}}\right)+\delta F_0\left(T,T_{\mathrm{mat}}\right)^2\ e^{\sigma^2 T},$$

and so, after simple algebra

$$\begin{aligned}\Delta\left(T,T_{\mathrm{mat}}\right)&=\mathsf{E}^{Q_{T_{\mathrm{mat}}}}\left[F\left(T,T_{\mathrm{mat}}\right)\ \frac{P_0\left(T,T_{\mathrm{mat}}\right)}{P\left(T,T_{\mathrm{mat}}\right)}\right]-F_0\left(T,T_{\mathrm{mat}}\right)\\&=F_0\left(T,T_{\mathrm{mat}}\right)\ \frac{\delta F_0\left(T,T_{\mathrm{mat}}\right)}{1+\delta F_0\left(T,T_{\mathrm{mat}}\right)}\ \left(e^{\sigma^2 T}-1\right).\end{aligned}$$

In summary,

$$\Delta\left(T,T_{\mathrm{mat}}\right)=F_0\left(T,T_{\mathrm{mat}}\right)\theta\ \left(e^{\sigma^2 T}-1\right)\ , \tag{3}$$

where

$$\theta=\frac{\delta F_0\left(T,T_{\mathrm{mat}}\right)}{1+\delta F_0\left(T,T_{\mathrm{mat}}\right)}\ .$$

Expanding the exponential to the first order, one can write the more familiar form for the convexity correction [2]:

$$\Delta\left(T,T_{\mathrm{mat}}\right)\simeq F_0\left(T,T_{\mathrm{mat}}\right)\theta\sigma^2 T. \tag{4}$$

The calculation above is an archetype for all approximate convexity computations and we will see it again.

3 CMS rates

The acronym CMS stands for *constant maturity swap*, and it refers to a swap rate which fixes in the future. CMS rates provide a convenient alternative to LIBOR as a floating index, as they allow market participants express their views on the future levels of *long term rates* (for example, the 10 year swap rate).

There are a variety of CMS based instruments the simplest of them being CMS swaps and CMS caps / floors. Valuation of these vanilla instruments will be the subject of the bulk of this lecture.

3.1 CMS swaps and caps / floors

A fixed for floating *CMS swap* is a periodic exchange of interest payments on a fixed notional in which the floating rate is indexed by a reference swap rate (say, the 10 year swap rate) rather than LIBOR. More specifically:

(a) The fixed leg pays a fixed coupon, quarterly, on the act/360 basis.

(b) The floating leg pays the 10 year[1] swap rate which fixes two business days before the start of each accrual period. The payments are quarterly on the act/360 basis and are made at the end of each accrual period.

A variation on a CMS swap is a LIBOR for CMS swap.

Note that using a swap rate as the floating rate makes this transaction a bit more difficult to price. Two things worth noting are:

(a) The floating leg of a CMS does not price at par! This has to do with the fact that the rate used in discounting over a 3 month period is the LIBOR rate and not the swap rate.

(b) In calculating the PV of the floating leg, we cannot use the forward swap rate as the future fixing of the swap rate, i.e. the CMS rate.

A *CMS cap* or *floor* is a basket of calls or puts on a swap rate of fixed tenor (say, 10 years) structured in analogy to a LIBOR cap or floor. For example, a 5 year cap on 10 year CMS struck at K is a basket of CMS caplets each of which:

(a) pays $\max(10 \text{ year CMS rate} - K, 0)$, where the CMS rate fixes two business days before the start of each accrual period;

(b) the payments are quarterly on the act/360 basis, and are made at the end of each accrual period.

The definition of a CMS floor is analogous.

3.2 Valuation of CMS swaps and caps / floors

Let us start with a single period $\left[T_{\text{start}},\ T_{\text{pay}}\right]$ CMS swap (a *swaplet*) whose fixed leg pays coupon C. Clearly, the PV of the fixed leg is

$$\mathsf{PV}_{\text{fixed}} = C\delta\, P\left(0, T_{\text{pay}}\right), \tag{5}$$

where δ is the coverage factor for the period $\left[T_{\text{start}},\ T_{\text{pay}}\right]$. The PV of the floating leg of the swaplet is

$$\mathsf{PV}_{\text{floating}} = P\left(0, T_{\text{pay}}\right)\delta\, \mathsf{E}^{\mathsf{Q}_{T_{\text{pay}}}}\left[S\left(T_{\text{start}}, T_{\text{mat}}\right)\right], \tag{6}$$

[1] Or whatever the tenor has been agreed upon.

where $Q_{T_{\text{pay}}}$ denotes the T_{pay}-forward martingale measure. Remember that T_{mat} denotes the maturity of the reference swap starting on T_{start}[2], and not the end of the accrual period. As a consequence,

$$\begin{aligned} \mathsf{PV}_{\text{CMS swaplet}} &= \mathsf{PV}_{\text{fixed}} - \mathsf{PV}_{\text{floating}} \\ &= P\left(0, T_{\text{pay}}\right) \delta \mathsf{E}^{\mathsf{Q}_{T_{\text{pay}}}} \left[C - S\left(T_{\text{start}}, T_{\text{mat}}\right)\right]. \end{aligned} \tag{7}$$

The PV of a CMS swap is obtained by summing up the contributions from all constituent swaplets.

The valuation of CMS caplets and floorlets is similar:

$$\mathsf{PV}_{\text{CMS caplet}} = P\left(0, T_{\text{pay}}\right) \delta \mathsf{E}^{\mathsf{Q}_{T_{\text{pay}}}} \left[\max\left(S\left(T_{\text{start}}, T_{\text{mat}}\right) - K, 0\right)\right] , \tag{8}$$

and

$$\mathsf{PV}_{\text{CMS floorlet}} = P\left(0, T_{\text{pay}}\right) \delta \mathsf{E}^{\mathsf{Q}_{T_{\text{pay}}}} \left[\max\left(K - S\left(T_{\text{start}}, T_{\text{mat}}\right), 0\right)\right] . \tag{9}$$

Not surprisingly, this implies a put / call parity relation for CMS: The PV of a CMS floorlet struck at K less the PV of a CMS caplet struck at the same K is equal to the PV of a CMS swaplet paying K. Let $C\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right)$ denote the break even CMS rate, given by

$$C\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right) = \mathsf{E}^{\mathsf{Q}_{T_{\text{pay}}}} \left[S\left(T_{\text{start}}, T_{\text{mat}}\right)\right] . \tag{10}$$

The notation is a bit involved, so let us be very specific.

(a) T_{start} denotes the start date of the reference swap (say, 1 year from now). This will also be the start of the accrual period of the swaplet.

(b) T_{mat} denotes the maturity date of the reference swap (say, 1o years from T_{start}).

(c) T_{pay} denotes the payment day on the swaplet (say, 3 months from T_{start}). This will also be the end of the accrual period of the swaplet.

In the name of completeness we should mention that one more date plays a role, namely the date on which the swap rate is fixed. This is usually two days before the start date, and we shall neglect its impact.

4 CMS convexity correction

The CMS rate is not a very intuitive concept! In this section we will express it in terms of more familiar quantities.

[2] Say, the 10 year anniversary of T_{start}.

4.1 The uses of Girsanov's theorem

Let $C\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right)$ denote the CMS rate given by (10). We shall write (10) in a more intuitive form. First, we apply Girsanov's theorem in order to change from the measure $\mathsf{Q}_{T_{\text{pay}}}$ to the measure Q associated with the annuity starting at T_{start}:

$$P_0\left(0, T_{\text{pay}}\right) \mathsf{E}^{\mathsf{Q}_{T_{\text{pay}}}}\left[S\left(T_{\text{start}}, T_{\text{mat}}\right)\right] = L_0\left(0\right) \mathsf{E}^{\mathsf{Q}}\left[\frac{S\left(T_{\text{start}}, T_{\text{mat}}\right) P\left(T_{\text{start}}, T_{\text{pay}}\right)}{L\left(T_{\text{start}}\right)}\right],$$

i.e.

$$\begin{aligned} C\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right) &= \mathsf{E}^{\mathsf{Q}_{T_{\text{pay}}}}\left[S\left(T_{\text{start}}, T_{\text{mat}}\right)\right] \\ &= \mathsf{E}^{\mathsf{Q}}\left[S\left(T_{\text{start}}, T_{\text{mat}}\right) \frac{L_0\left(T_{\text{start}}\right)}{L\left(T_{\text{start}}\right)} \frac{P\left(T_{\text{start}}, T_{\text{pay}}\right)}{P_0\left(T_{\text{start}}, T_{\text{pay}}\right)}\right]. \end{aligned} \tag{11}$$

This formula looks complicated! However, it has the advantage of being expressed in terms of the "natural" martingale measure.

We write

$$\frac{L_0\left(T_{\text{start}}\right)}{L\left(T_{\text{start}}\right)} \frac{P\left(T_{\text{start}}, T_{\text{pay}}\right)}{P_0\left(T_{\text{start}}, T_{\text{pay}}\right)} = 1 + \left(\frac{L_0\left(T_{\text{start}}\right)}{L\left(T_{\text{start}}\right)} \frac{P\left(T_{\text{start}}, T_{\text{pay}}\right)}{P_0\left(T_{\text{start}}, T_{\text{pay}}\right)} - 1\right),$$

and notice that, by the martingale property of the annuity measure,

$$\mathsf{E}^{\mathsf{Q}}\left[S\left(T_{\text{start}}, T_{\text{mat}}\right)\right] = S\left(T_{\text{start}}, T_{\text{mat}}\right),$$

the current value of the forward swap rate! As a result,

$$\begin{aligned} & C\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right) \\ &\quad = S\left(T_{\text{start}}, T_{\text{mat}}\right) + \mathsf{E}^{\mathsf{Q}}\left[S\left(T_{\text{start}}, T_{\text{mat}}\right)\left(\frac{L_0\left(T_{\text{start}}\right)}{L\left(T_{\text{start}}\right)} \frac{P\left(T_{\text{start}}, T_{\text{pay}}\right)}{P_0\left(T_{\text{start}}, T_{\text{pay}}\right)} - 1\right)\right] \\ &\quad = S\left(T_{\text{start}}, T_{\text{mat}}\right) + \Delta\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right), \end{aligned}$$

where $\Delta\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right)$ denotes the *CMS convexity correction*, i.e. the difference between the forward swap rate and the CMS rate.

The CMS convexity correction can be attributed to two factors:

(a) Intrinsics of the dynamics of the swap rate which we shall, somewhat misleadingly, delegate to the correlation effects between LIBOR and swap rate.

(b) Payment delay.

Correspondingly, we have the decomposition:

$$\Delta\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right) = \Delta_{\text{corr}}\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right) + \Delta_{\text{delay}}\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right), \quad (12)$$

which is obtained by substituting the identity

$$\begin{aligned} &\frac{L_0\left(T_{\text{start}}\right)}{L\left(T_{\text{start}}\right)} \frac{P\left(T_{\text{start}}, T_{\text{pay}}\right)}{P_0\left(T_{\text{start}}, T_{\text{pay}}\right)} - 1 \\ &\quad = \left(\frac{L_0\left(T_{\text{start}}\right)}{L\left(T_{\text{start}}\right)} - 1\right) + \frac{L_0\left(T_{\text{start}}\right)}{L\left(T_{\text{start}}\right)} \left(\frac{P\left(T_{\text{start}}, T_{\text{pay}}\right)}{P_0\left(T_{\text{start}}, T_{\text{pay}}\right)} - 1\right). \end{aligned}$$

into the representation (11) of the CMS convexity correction. Explicitly,

$$\Delta_{\text{corr}}\left(T_{\text{start}}, T_{\text{mat}}\right) = \mathsf{E}^{\mathsf{Q}}\left[S\left(T_{\text{start}}, T_{\text{mat}}\right)\left(\frac{L_0\left(T_{\text{start}}\right)}{L\left(T_{\text{start}}\right)} - 1\right)\right], \quad (13)$$

and

$$\begin{aligned} &\Delta_{\text{delay}}\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right) \\ &\quad = \mathsf{E}^{\mathsf{Q}}\left[S\left(T_{\text{start}}, T_{\text{mat}}\right) \frac{L_0\left(T_{\text{start}}\right)}{L\left(T_{\text{start}}\right)} \left(\frac{P\left(T_{\text{start}}, T_{\text{pay}}\right)}{P_0\left(T_{\text{start}}, T_{\text{pay}}\right)} - 1\right)\right]. \end{aligned} \quad (14)$$

Note that $\Delta_{\text{delay}}\left(T_{\text{start}}, T_{\text{mat}} \middle| T_{\text{pay}}\right)$ is zero, if the CMS rate is paid at the beginning of the accrual period.

4.2 Calculating the CMS convexity correction

The formulas for the CMS convexity adjustments derived above are model independent, and one has to make choices in order to produce workable numbers. The issue of accurate calculation of the CMS corrections derived in the previous subsection has been the subject of intensive research. The difficulty lies, of course, in our ignorance about the details of the martingale measure Q. Among the proposed approaches we list the following:

(a) Black model style calculation. This is the method that we will use below.

(b) Replication method. This method attempts to replicate the payoff of a CMS caplet / floorlet by means of European swaptions of various strikes. This method allows one to take the volatility smile effects into account by, say, using the SABR model.

(c) Use Monte Carlo simulation in conjunction with a term structure model[3] This method is somewhat slow and its success depends on the accuracy of the term structure model.

Let us explain method (a) which, while not perfectly accurate, gives a respectable result for magnitude of the correction. In order to produce a closed form result, we shall make the following crude approximations:

(a) All day count fractions are equal to $1/f$, where f is the frequency of payments on the reference swap (typically, $f = 2$).

(b) All discounting is in terms of a single forward swap rate S. The level function is thus given by

$$\begin{aligned} L(t) &= \frac{1}{f} \sum_{j=1}^{n} \frac{1}{(1 + S(t)/f)^j} \\ &= \frac{1}{S(t)} \left(1 - \frac{1}{(1 + S(t)/f)^n}\right), \end{aligned} \tag{15}$$

and the discount factor from the start date to the payment date is

$$\begin{aligned} P(t) &= \frac{1}{1 + S(t)/f_{\text{CMS}}} \\ &\simeq \frac{1}{(1 + S(t)/f)^{f/f_{\text{CMS}}}}, \end{aligned} \tag{16}$$

where f_{CMS} is the frequency of payments on the CMS swap (typically $f_{\text{CMS}} = 4$).

(c) The swap rate follows the Black model dynamics.

In order to lighten up on the notation, we suppress the function arguments in the following calculation. We begin by Taylor expanding $1/L$ in powers of S around S_0:

$$\begin{aligned} \frac{1}{L} &\simeq \frac{1}{L_0} + \frac{d}{dS_0}\left(\frac{1}{L_0}\right)(S - S_0) \\ &= \frac{1}{L_0}\left(1 + \frac{1}{S_0}\left(1 - \frac{1}{1 + S_0/f}\frac{nS_0/f}{(1 + S_0/f)^n - 1}\right)(S - S_0)\right) \\ &\equiv \frac{1}{L_0}\left(1 + \theta_c \, \frac{S - S_0}{S_0}\right). \end{aligned}$$

[3] We shall discuss term structure models in the following lectures.

Assuming Black's model for the swap rate,

$$S(t) = S_0 e^{\sigma W(t) - \frac{1}{2}\sigma^2 t},$$

and using (2) we this find that

$$\mathsf{E}^{\mathsf{Q}}\left[S\,\frac{L_0}{L}\right] = S_0 + S_0\theta_c\left(e^{\sigma^2 T} - 1\right).$$

Similarly,

$$\mathsf{E}^{\mathsf{Q}}\left[S\,\frac{L_0}{L}\left(\frac{P}{P_0} - 1\right)\right] \simeq -S_0\theta_d\left(e^{\sigma^2 T} - 1\right),$$

where

$$\theta_d = \frac{S_0/f_{\text{CMS}}}{1 + S_0/f}\ .$$

Using (2), and reinstating the arguments we find the following expressions for the convexity corrections:

$$\begin{aligned} \Delta_{\text{corr}}\left(T, T_{\text{mat}} \middle| T_{\text{pay}}\right) &\simeq S_0\left(T, T_{\text{mat}}\right)\theta_c\left(e^{\sigma^2 T} - 1\right), \\ \Delta_{\text{delay}}\left(T, T_{\text{mat}} \middle| T_{\text{pay}}\right) &\simeq -S_0\left(T, T_{\text{mat}}\right)\theta_d\left(e^{\sigma^2 T} - 1\right). \end{aligned} \tag{17}$$

These are our approximate expressions for the CMS convexity corrections.

For the record, let us make the following two observations: On can combine the impact of correlations and payment delay into one formula,

$$\theta_c - \theta_d = 1 - \frac{S_0/f}{1 + S_0/f}\left(\frac{f}{f_{\text{CMS}}} + \frac{n}{\left(1 + S_0/f\right)^n - 1}\right). \tag{18}$$

Expanding the exponential, the convexity adjustment can be written in the more traditional form:

$$\Delta\left(T, T_{\text{mat}} \middle| T_{\text{pay}}\right) \simeq S_0\left(T, T_{\text{mat}}\right)\left(\theta_c - \theta_d\right)\sigma^2 T. \tag{19}$$

5 Eurodollar futures / FRAs convexity corrections

The final example of a convexity correction is that between a Eurodollar future and a FRA. What is its financial origin? Consider an investor with a long position in a Eurodollar contract.

1. A FRA does not have any intermediate cash flows, while Eurodollar futures are marked to market by the Exchange daily. This means daily cash flows in and out of the margin account. The implication for the investor's P&L is that it is negatively correlated with the dynamics of interest rates: If rates go up, the price of the contract goes down, and the investor needs to add money into the margin account, rather than investing it at higher rates (opportunity loss for the investor). If rates go down, the contract's price goes up, and the investor withdraws money out of the margin account and invests at a lower rate (opportunity loss for the investor again). The investor should thus demand a discount on the contract's price in order to be compensated for these adverse characteristics of his position compare to being long a FRA. As a result, the LIBOR calculated from the price of a Eurodollar futures contract has to be higher than the corresponding LIBOR forward.

2. A Eurodollar future is cash settled at maturity (rather than at the end of the accrual period). The investor should be compensated by a lower price. This effect is analogous to the payment delay we discussed in the context of LIBOR in arrears and is relatively small.

Mathematically, because of the daily mark to market, the appropriate measure defining the Eurodollar future is the spot measure Q_0. From Girsanov's theorem we obtain the following equation for the Eurodollar future implied LIBOR:

$$\mathsf{E}^{\mathsf{Q}_0}\left[F\left(T,T_{\text{mat}}\right)\right]=\mathsf{E}^{\mathsf{Q}_{T_{\text{mat}}}}\left[F\left(T,T_{\text{mat}}\right)\ \frac{P_0\left(0,T_{\text{mat}}\right)}{B_0\left(T\right)P\left(T,T_{\text{mat}}\right)}\right], \tag{20}$$

where $B\left(t\right)$ is the price of the rolling bank account. The ED / FRA convexity correction is thus given by

$$\Delta_{\text{ED / FRA}}\left(T,T_{\text{mat}}\right)=\mathsf{E}^{\mathsf{Q}_{T_{\text{mat}}}}\left[F\left(T,T_{\text{mat}}\right)\left(\frac{P_0\left(0,T_{\text{mat}}\right)}{B_0\left(T\right)P\left(T,T_{\text{mat}}\right)}-1\right)\right]. \tag{21}$$

In order to derive a workable numerical value for $\Delta_{\text{ED / FRA}}\left(T,T_{\text{mat}}\right)$, it is best to use a short rate term structure model. We will discuss this in the next lecture.

References

[1] Brigo, D., and Mercurio, F.: *Interest Rate Models - Theory and Practice*, Springer Verlag (2006).

[2] Hull, J.: *Options, Futures and Other Derivatives* Prentice Hall (2005).