1. The forward curve

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1 LIBOR and LIBOR based instruments

LIBOR (= London Interbank Offered Rate) is the interest rate at which banks borrow large amounts of money from each other. It is a widely used benchmark for short term (overnight to 1 year) interest rates. Daily *fixings* of LIBOR are published by the British Banking Association on each London business day at 11 a.m.

London time. These fixings are calculated from quotes provided by a number of participating banks. LIBOR is not a risk free rate, but it is close to it: the participating banks have high credit ratings.

LIBOR is offered in ten major currencies: GBP, USD, EUR, JPY, CHF, CAD, AUD, DKK, SED, and NZD. Throughout this course we shall assume a single currency, namely the USD.

In the USD, LIBOR applies to deposits that begin two business days from the current date (this is called the *spot date*) and whose maturity is on an anniversary date (say, 3 months) of that settlement date. Determining the anniversary date follows two rules:

(a) If the anniversary date is not a business day, move forward to the next business day, except if this takes you over a calendar month end, in which case you move back to the last business day. This rule is known as *modified following business day convention*.

(b) If the settlement date is the last business day of a calendar month, all anniversary dates are last business days of their calendar months.

In addition to spot transactions, there are a variety of *vanilla* LIBOR based instruments actively trading both on exchanges and over the counter: LIBOR futures, forward rate agreements. The markets for LIBOR based instruments are among the world's largest financial markets. The significance of these instruments is that:

(a) They allow portfolio managers and other financial professionals effectively hedge their interest rates exposure.

(b) One can use them to synthetically create desired future cash flows and thus effectively manage assets versus liabilities.

(c) They allow market participants easily express their views on future levels of interest rates.

1.1 Forward rate agreements

Forward rate agreements (FRAs) are over the counter (OTC) instruments. In a FRA transaction, one of the counterparties (A) agrees to pay the other counterparty (B) LIBOR settling t years from now applied to a certain notional amount (say, $\$100$mm. In exchange, counterparty B pays counterparty A a pre-agreed interest rate (say, 3.05%) applied to the same notional. The contract matures on an anniversary T (say, 3 months) of the settlement date, and interest is computed on an act/360 day count basis. Anniversary dates generally follow the same modified following business day convention as the LIBOR. FRAs are quoted in terms of the annualized forward interest rate applied to the accrual period of the transaction.

1.2 LIBOR futures

LIBOR futures (known also as the *Eurodollar futures*) are exchange traded futures contracts (they trade on the Chicago Mercantile Exchange) on the 3 month LIBOR rate. They are similar to FRAs, except that their terms (such as maturity dates) are regulated by the exchange. Each of the contracts assumes the notional principal of \$1,000,00. Interest on these contracts is computed on an act/360 day count basis. Eurodollar futures are structured so that a single contract pays \$25 for each 1 basis point movement in LIBOR. The market convention is to quote the rates R on the Eurodollar futures in terms of the "price" defined as

$$100 \times (1 - R)\,.$$

Consequently, Eurodollar futures quotes are linear in interest rates, unlike LIBOR deposits, FRAs, and swaps (described below) which are non-linear ("convex") in interest rates. We shall return to this point in Lecture 3.

At any time, 44 Eurodollar contracts are listed:

- 40 *quarterly* contracts maturing on the third Wednesday of the months March, June, September, and December over the next 10 years. Of these contracts, only the first 20 are liquid, the open interest in the remaining 20 being minimal. Their maturity dates are the 3 month anniversary dates of these value dates. As it happens, the third Wednesday of a month has the convenient characteristic that it is never a New York or London holiday and its anniversary dates are always good business days.

- 4 *serial contracts* maturing on the third Wednesday of the nearest four months not covered by the above quarterly contracts. Of these 4 contracts, typically the first two are liquid.

1.3 Swaps

A (fixed for floating) *swap* is an OTC transaction in which two counterparties agree to exchange periodic interest payments on a prespecified *notional* amount. One counterparty (the fixed *payer*) agrees to pay periodically the other counterparty (the fixed *receiver*) a fixed coupon (say, 5.35% per annum) in exchange for receiving periodic LIBOR applied to the same notional.

Spot starting swaps based on LIBOR begin on a start date 2 business days from the current date and mature and pay interest on anniversary dates that use the same modified following business day conventions as the LIBOR index. Interest is usually computed on an act/360 day basis on the floating side of the swap and on 30/360 day basis in the fixed side of the pay. Typically, fixed payment dates ("coupon dates") are semiannual (every 6 months), and floating payment dates are quarterly (every 3 months) to correspond to a 3 month LIBOR. In addition to spot starting swaps, *forward starting* swaps are routinely traded. In a forward starting swap, the first accrual period can be any business day beyond spot. Swaps (spot and forward starting) are quoted in terms of the fixed coupon.

2 Valuation of LIBOR based instruments

In this lecture we are concerned with valuation and risk management of non-contingent (but not necessarily known) future cash flows. The building blocks required are:

(a) *Discount factors*, which allow one to calculate present value of money received in the future.

(b) *Forward rates*, which allow one to make assumptions as to the future levels of rates.

2.1 Zero coupon bonds

A *zero coupon bond* (or *discount bond*) for maturity T is an instrument which pays $\$1$ T years from now. We denote its market value by $P(0,T) > 0$. It is thus the present value (abbreviated PV) of $\$1$ guaranteed to be paid at time T. The market does not contain enough information in order to determine the prices of zero coupon bonds for all values of T, and arbitrary choices have to be made. Later in this lecture we will discuss how to do this in ways that are consistent with all the available information. In the meantime, we will be using these prices in order to calculate present values of future cash flows (both guaranteed and contingent), and refer to $P(0,T)$ as the *discount factor* for time T.

Consider a *forward contract* on a zero coupon bond: at some future time $t < T$, we deliver to the counterparty $\$1$ of a zero coupon bond of final maturity T. What is the fair price $P(t,T)$ paid at delivery? We calculate it using the following no arbitrage argument which provides a risk-free replication of the forward trade in terms of spot trades.

1. We buy $\$1$ of a zero coupon bond of maturity T today for the price of $P(0,T)$.

2. We finance this purchase by short selling a zero coupon bond of maturity t and notional $P(0,T)/P(0,t)$ for overall zero initial cost.

3. In order to make the trade self-financing, we need to charge this amount at delivery. Thus,
$$P(t,T) = \frac{P(0,T)}{P(0,t)} \,. \tag{1}$$

The forward price $P(t,T)$ is also called the (forward) *discount factor* for maturity T and value date t.

Two important facts about discount factors are[1]:

(a)
$$P(t,T) < 1, \tag{2}$$
i.e. the value of a dollar in the future is less than the its value now.

(b)
$$\frac{\partial P(t,T)}{\partial T} < 0, \tag{3}$$
which means that the future value of a dollar decreases as the payment date gets pushed further away.

2.2 Valuation of FRAs and forward rates

Discount factors can be expressed in terms of interest rates. A convenient, albeit purely theoretical concept is that of the continuously compounded *instantaneous forward rate* $f(t)$. In terms of $f(t)$,

$$P(t,T) = \exp\left(-\int_t^T f(s)\,ds\right). \tag{4}$$

This equation is merely the definition of $f(t)$, and expresses the discount factor as the result of continuous discounting of the value of a dollar between the value and maturity dates.

Conversely, the instantaneous forward rate can be computed from the discount factor:

$$\begin{aligned} f(t) &= -\frac{1}{P(t,T)}\,\frac{\partial P(t,T)}{\partial T}\Big|_{T=t} \\ &= -\frac{\partial}{\partial T}\,\log P(t,T)\,\Big|_{T=t}. \end{aligned} \tag{5}$$

[1] In some markets, these properties are known to have been violated.

The forward rate $F(t,T)$ for the time t and maturity T is defined as the (annual) interest rate on a FRA starting at t and ending at T. This is the pre-agreed fixed interest rate on a FRA contract. In order to compute it, let δ denote the day count factor for the period spanned by the FRA. Then,

$$P(t,T) = \frac{1}{1+\delta F(t,T)} \ ,$$

and thus

$$\begin{aligned} F(t,T) &= \frac{1}{\delta}\ \left(\frac{1}{P(t,T)} - 1\right) \\ &= \frac{1}{\delta}\left(\exp\int_t^T f(s)\,ds - 1\right). \end{aligned} \tag{6}$$

Econometric studies of historical rates data show that forward rates are poor predictors of future interest rates. Rather, they reflect the evolving current consensus market sentiment about the future levels of rates. Their true economic significance lies in the fact that a variety of instruments whose values derive from the levels of forward rates (such as swaps) can be liquidly traded and used to hedge against adverse future levels of rates.

2.3 Valuation of swaps and swap rates

We first consider a spot starting swap. Let $T_1 < \ldots < T_{n_{\text{fixed}}}$ denote the coupon dates of the swap, and let $T_0 = 0$. The PV of the interest payments on the fixed leg of a swap is calculated by adding up the PVs of all future cash flows:

$$\mathsf{PV}_{\text{fixed}} = \sum_{j=1}^{n_{\text{fixed}}} \alpha_j C P(0,T_j)\,, \tag{7}$$

where C is the coupon rate, $P(0,T_j)$ are the discount factors, and α_j are the day count fractions applying to each semi-annual period (the number of days based on a 30/360 day count divided by 360). It is useful to write this formula as

$$\mathsf{PV}_{\text{fixed}} = CL, \tag{8}$$

where

$$L = \sum_{j=1}^{n_{\text{fixed}}} \alpha_j P(0,T_j)\,, \tag{9}$$

is called the *level* (or the *DVO1*) of the swap.

For the floating leg, the valuation formula reads:

$$\mathsf{PV}_{\text{floating}} = \sum_{j=1}^{n_{\text{float}}} \delta_j L_j P\left(0, T_j\right), \tag{10}$$

where

$$\begin{aligned} L_j &= F\left(T_{j-1}, T_j\right) \\ &= \frac{1}{\delta_j}\left(\frac{1}{P\left(T_{j-1}, T_j\right)} - 1\right) \end{aligned} \tag{11}$$

is the 3 month LIBOR forward rate for settlement at T_{j-1}, $P\left(0, T_j\right)$ (here $T_0 = 0$) is the discount factor and δ_j is the day count fraction applying to each quarterly period (the number of days based on a act/360 day count divided by 360).

An important fact about swap valuation is that

$$\mathsf{PV}_{\text{floating}} = 1 - P\left(0, T_{\text{mat}}\right), \tag{12}$$

where T_{mat} denotes the maturity of the swap. This equation, stated as

$$\mathsf{PV}_{\text{floating}} + P\left(0, T_{\text{mat}}\right) = 1,$$

expresses the fact that a spot settled floating rate bond, paying LIBOR and repaying the principal at maturity, is always valued at par[2]. The proof of (12) is straightforward:

$$\begin{aligned} \mathsf{PV}_{\text{floating}} &= \sum_{j=1}^{n_{\text{float}}} \delta_j L_j P\left(0, T_j\right) \\ &= \sum_{j=1}^{n_{\text{float}}} \left(\frac{1}{P\left(T_{j-1}, T_j\right)} - 1\right) P\left(0, T_j\right) \\ &= \sum_{j=1}^{n_{\text{float}}} \left(P\left(0, T_{j-1}\right) - P\left(0, T_j\right)\right) \\ &= 1 - P\left(0, T_{n_{\text{float}}}\right). \end{aligned}$$

The PV of a swap is the difference between the PVs of the fixed and floating legs (in this order!):

$$\mathsf{PV}_{\text{swap}} = \mathsf{PV}_{\text{fixed}} - \mathsf{PV}_{\text{floating}}.$$

[2]This is not strictly true once LIBOR has been fixed, as in a seasoned swap.

A break-even (or mid-market) swap has zero PV:

$$\mathsf{PV}_{\text{fixed}} = \mathsf{PV}_{\text{floating}}.$$

That uniquely determines the coupon on a mid-market swap:

$$S\left(T_{\text{mat}}\right) = \frac{1 - P\left(0, T_{\text{mat}}\right)}{L}, \tag{13}$$

called the (mid-market) *swap rate*.

2.4 Valuation of forward starting swaps

Valuation of forward starting swaps is similar to the valuation of spot starting swaps. Let $T_1 < \ldots < T_{n_{\text{fixed}}}$ denote the coupon dates of the swap, and let $T_0 = T_{\text{start}} > 0$ denote the settlement date of the swap. The basic property of the floating leg of a swap reads now:

$$\mathsf{PV}_{\text{floating}} = P\left(0, T_{\text{start}}\right) - P\left(0, T_{\text{mat}}\right). \tag{14}$$

The coupon on a break-even swap is now

$$S\left(T_{\text{start}}, T_{\text{mat}}\right) = \frac{P\left(0, T_{\text{start}}\right) - P\left(0, T_{\text{mat}}\right)}{L}, \tag{15}$$

where the level function of the forward starting swap is again given by (9).

It is instructive to rewrite this equation as

$$S\left(T_{\text{start}}, T_{\text{mat}}\right) = \frac{1 - P\left(T_{\text{start}}, T_{\text{mat}}\right)}{L\left(T_{\text{start}}\right)}, \tag{16}$$

where the forward level function is now given by

$$L\left(T_{\text{start}}\right) = \sum_{j=1}^{n_{\text{fixed}}} \alpha_j P\left(T_{\text{start}}, T_j\right). \tag{17}$$

It means that the forward swap rate is given by the same expression as the spot swap rate with the discount factors replaced by the forward discount factors!

3 Building a LIBOR forward curve

So far we have been assuming that all discount factors $P\left(t, T\right)$, or equivalently, all forward rates $F\left(t, T\right)$ are known. Now we will discuss the methods of calculating these quantities from the available market information. The result can be presented in the various equivalent forms:

(a) As a function $t \to F(t, T)$ with fixed tenor $T - t$ (say, $T - t = 3$ months). This is called the *forward curve*.

(b) As a function $T \to P(0, T)$. This is called the *discount curve* (or *zero coupon curve*).

(c) As a collection of spot starting swap rates for all tenors. This is called the *par swap curve*.

The curve construction should be based on the prices of liquidly traded benchmark securities. As this set of securities is incomplete, we need a robust and efficient method involving interpolation and, if necessary, extrapolation. These benchmark instruments include deposit rates, Eurodollar futures and a number of benchmark swaps. Benchmark swaps are typically spot starting, and have maturities from 1 year to 40 years and share the same set of coupon dates. For example, one could use the following set of instruments:

(a) Overnight, 1 week, 2 week, 1 month, 2 month, and 3 month deposit rates.

(b) The first 8 Eurodollar contracts.

(c) Spot starting swaps with maturities 2, 3, 4, 5, 7, 10, 12, 15, 20, 25, and 30 years.

3.1 Bootstrapping techniques

The standard (and oldest) method for building a LIBOR forward curve uses *bootstrapping*, and consists in the following. Suppose that we know the discount factors

$$P(0, T_j), \qquad j = 1, \ldots, N, \tag{18}$$

for all "standard" maturities T_j spaced (say) every 3 months. It is important to choose these maturities so that they include the coupon dates of the benchmark swaps. Then,

$$P(T_{j-1}, T_j) = \frac{P(0, T_j)}{P(0, T_{j-1})},$$

and so we we can calculate the forward rates for all standard maturities:

$$F(T_{j-1}, T_j) = \frac{1}{\delta_j}\left(\frac{1}{P(T_{j-1}, T_j)} - 1\right).$$

That does not really solve the problem yet, because we are now faced with the issue of computing the forward rates for non-standard settlements T (say, a 3 month forward settling 4 months from now). We compute these forwards by means of *interpolation*. There is no standard for interpolation and various schemes have been proposed. Here is a partial list:

(a) *Linear interpolation of the discount factors*:

$$P(0,T) = \frac{T_j - T}{T_j - T_{j-1}}\, P(0,T_{j-1}) + \frac{T - T_{j-1}}{T_j - T_{j-1}}\, P(0,T_j)\,,$$

for $T_{j-1} \leq T \leq T_j$.

(b) *Linear interpolation of the log discount factors*:

$$\log P(0,T) = \frac{T_j - T}{T_j - T_{j-1}}\, \log P(0,T_{j-1}) + \frac{T - T_{j-1}}{T_j - T_{j-1}}\, \log P(0,T_j)\,,$$

for $T_{j-1} \leq T \leq T_j$.

(c) *Constant instantaneous forward rate*. We assume that $f(t) = f_j = \text{const}$, i.e.

$$P(T_{j-1},T_j) = \exp\left(-f_j \times (T_j - T_{j-1})\right).$$

This implies that

$$f_j = -\frac{1}{T_j - T_{j-1}}\, \log P(T_{j-1},T_j)\,,$$

for all j, and we can now easily carry out the integration $\int_t^T f(s)\,ds$ in the definition of $P(t,T)$ with arbitrary t and T.

(d) *Linear instantaneous forward rate*. Instantaneous forward rates are assumed linear between the benchmark maturities and continuous throughout. This is a refinement of scheme (c) which requires matching the values of the instantaneous rate at the benchmark maturities.

(e) *Quadratic instantaneous forward rate*. Instantaneous forward rates are assumed quadratic between the benchmark maturities and continuously once differentiable throughout. This is a further refinement of scheme (c) which requires matching the values *and* the first derivatives of the instantaneous rate at the benchmark maturities.

How do we determine the discount factors (18) for the standard maturities? This usually proceeds in three steps:

(a) Build the short end (approximately, the first 3 months) of the curve using LIBOR deposit rates and, possibly, some Eurodollar futures[3]. This step will involve some interpolation.

(b) Build the intermediate (somewhere between 3 months and 5 years) part of the curve using the (convexity-adjusted) Eurodollar futures. The starting date for the first future has its discount rate set by interpolation from the already built short end of the curve. With the addition of each consecutive future contract to the curve the discount factor for its starting date is either (a) interpolated from the existing curve if it starts earlier than the end date of the last contract, or (b) extrapolated from the end date of the previous future. Any of the interpolation schemes described above can be used.

(c) Build the long end of the curve using swap rates as par coupon rates. Observe first that for a swap of maturity T_{mat} we can calculate the discount factor $P(0, T_{\text{mat}})$ in terms of the discount factors to the earlier coupon dates:

$$P(0, T_{\text{mat}}) = \frac{1 - S(T_{\text{mat}}) \sum_{j=1}^{n-1} \alpha_j P(0, T_j)}{1 + \alpha_n S(T_{\text{mat}})} .$$

We begin by interpolating the discount factors for coupon dates that fall within the previously built segment of the curve, and continue by inductively applying the above formula. The problem is, of course, that we do not have market data for swaps with maturities falling on all standard dates (benchmark swaps have typically maturities 2 years, 3 years, 4 years, 5 years,...) and interpolation is again necessary to deal with the intermediate dates.

With regard to step (c) above we should mention that it is not a good idea to linearly interpolate par swap rates of different maturities (say, interpolate the 10 year rate and the 30 year rate in order to compute the 19 year rate). A better approach is to use one of the instantaneous forward rate interpolation schemes.

3.2 Smoothing B-splines fitting

In this approach, we work directly with the instantaneous forward rate $f(t)$ which we represent as a cubic B-spline (see the Appendix for the definition and properties of B-splines). We assume that the curve starts at $T_0 = 0$ and ends at T_{max} (say, 30 years), and choose K knot points $t_{-3}, \ldots, t_{N+4}$, with

$$t_{-3} < \ldots < t_0 = 0 < t_1 < \ldots < t_N = T_{max} < \ldots < t_{N+4},$$

[3]This will certainly be true, if the front contract is close to expiration, and if one decides to include the serial contracts into the benchmark instruments.

and let $B_k(t) \equiv B_k^{(3)}(t)$, $k = -3, -2, \ldots$, be the k-the basis function corresponding to these knot points. We represent $f(t)$ as a linear combination of the basis functions:

$$f(t) = \sum_{k=-3}^{N+4} f_k B_k(t) . \tag{19}$$

Note that, in this representation, the discount factors are simple functions of the f_k's:

$$P(t,T) = \exp\left(-\sum_{k=-3}^{N+4} \gamma_k(t,T) f_k\right), \tag{20}$$

where the coefficients

$$\gamma_k(t,T) = \int_t^T B_k(t)\, dt \tag{21}$$

can be easily computed using the algorithm presented in the Appendix.

Our goal is to choose the coefficients f_k in (19) consistently with the market data. This will be achieved by minimizing a suitable objective function. Suppose now that we are given a number of benchmark rates:

(a) Deposit rates $D_1 = F(0,T_1), \ldots, F_m = F(0,T_m)$, whose current market values are $\overline{D}_1, \ldots \overline{D}_m$.

(b) Forward rates $F_1 = F(t_1,T_1), \ldots, F_n = F(t_n,T_n)$, whose current market values are $\overline{F}_1, \ldots \overline{F}_n$. The tenors of the different rates (t_j, T_j) may overlap with each other and the tenors of the deposit and swap rates.

(c) Swap rates $S_1, \ldots, S_p$, whose current market values are $\overline{S}_1, \ldots \overline{S}_p$.

As a consequence of (20), all these rates are simple and explicit functions of the f_k's. For example, a forward rate is written as

$$F(t,T) = \frac{1}{\delta}\left(\exp\left(\sum_{k=-3}^{N+4} \gamma_k(t,T) f_k\right) - 1\right).$$

Denote the benchmark rates by $R_1, \ldots, R_{m+n+p}$, and consider the following objective function:

$$Q(f_{-3}, \ldots, f_{N+4}) = \frac{1}{2}\sum_{j=1}^{m+n+p} (R_j - \overline{R}_j)^2 + \frac{1}{2}\,\lambda \int_{T_0}^{T_{max}} f''(t)^2\, dt, \tag{22}$$

where λ is a non-negative constant. The second term on the right hand side of (3.2) is a *Tikhonov regularizer*, and its purpose is to penalize the "wiggliness" of

$f(t)$ at the expense of the accuracy of the fit. Its magnitude is determined by the magnitude of λ: the bigger the value of λ, the smoother the instantaneous forward rate at the expense of the fit. One may choose to refine the Tikhonov regularizer by replacing it with

$$\int_{T_0}^{T_{max}} \lambda(t) f''(t)^2 \, dt,$$

where $\lambda(t)$ is a non-negative (usually, piecewise constant) function. Experience shows that it is a good idea to choose $\lambda(t)$ smaller in the short end and larger in the back end of the curve.

The minimum of (3.2) can be found by means of standard Newton-type optimization algorithms such as the Levenberg-Marquardt algorithm (see e.g. [4]). The Levenberg-Marquardt algorithm applies to an objective function which can be written as a sum of squares of "residuals". This algorithm requires explicit formulas for the partial derivatives of the residuals with respect to the parameters of the problem. In our case, these derivatives can be readily computed.

3.3 Pros and cons of the two methods

Both methods explained above have their advantages and disadvantages. For the bootstrapping method, the list of pros and cons includes:

(a) Simplicity, bootstrapping does not require using optimization algorithms, all calculations are essentially done in closed form.

(b) Calculated swap rates fit exactly the benchmark swap rates.

(c) It is difficult to fit the short end of the curve where many instruments with overlapping tenors exist.

(d) Some of the interpolation schemes lead to saw-toothed shaped forwards which may lead to unstable pricing.

(e) The forward curve tends to be wiggly.

The list of pros and cons for the smoothing B-splines fitting method includes:

(a) The method requires optimization, and thus is slightly slower.

(b) Calculated swap rates are very close, but typically not equal, to the benchmark swap rates.

(c) There is no issue with overlapping tenors on instruments in the short end.

(d) The forward curve is smooth.

Other considerations, such as suitability for risk management, will be discussed later.

4 Curve sensitivities and position hedging

A common problem faced by portfolio managers and traders is to hedge the interest rate exposure of a portfolio of fixed income securities such as bonds, swaps, options, etc. The key issue is to quantify this exposure and offset it (if desired) by taking positions in liquid vanilla instruments. We let Π denote this portfolio, whose details are not important to us right now.

4.1 Input perturbation sensitivities

In this approach we compute the sensitivities of the portfolio to the benchmark instruments used in the curve construction, and replicate the risk of the portfolio by means of a portfolio consisting of the suitably weighted benchmark instruments.

(a) Compute the partial DVO1s of the portfolio Π to each of the benchmark instruments B_i: We shift each of the benchmark rates down 1 bp and calculate the corresponding changes $\delta_i \Pi$ in the PV.

(b) Compute the DVO1s $\delta_i B_i$ of the PVs of the benchmark instruments under these shifts.

(c) The hedge ratios Δ_i of the portfolio to the benchmarks are given by:

$$\Delta_i = \frac{\delta_i \Pi}{\delta_i B_i} \, .$$

This way of computing portfolio risk works well together with the bootstrapping method of building the curve.

4.2 Regression based sensitivities

An alternative and more robust approach consists in computing the sensitivities of the portfolio to a number of virtual scenarios, and expressing these sensitivities in terms of the sensitivities of a suitably selected hedging portfolio. We proceed as follows.

First, we select the hedging portfolio and the scenarios. This should be done judiciously, based on the understanding of the risks of the portfolio and liquidity of instruments intended as hedges.

(a) Choose a "hedging portfolio" consisting of vanilla instruments such as (spot or forward starting) swaps, Eurodollar futures, forward rate agreements, etc:

$$\Pi_{\text{hedge}} = \{B_1, \ldots, B_n\} \ .$$

(b) Let $\mathcal{C}_0$ denote the current forward curve (the "base scenario"). Choose a number of new micro scenarios

$$\mathcal{C}_1, \ldots, \mathcal{C}_p$$

by perturbing a segment of $\mathcal{C}_0$. For example, $\mathcal{C}_1$ could result from $\mathcal{C}_0$ by shifting the first 3 month segment down by 1 bp.

We then compute the sensitivities of the portfolio and the hedging portfolio under these curve shifts:

(a) The vector $\delta\Pi$ of portfolio's *sensitivities* under these scenarios is

$$\delta_i \Pi = \Pi\left(\mathcal{C}_i\right) - \Pi\left(\mathcal{C}_0\right), \qquad i = 1, \ldots, p,$$

where by $\Pi\left(\mathcal{C}_i\right)$ we denote the value of the portfolio given the shifted forward curve $\mathcal{C}_i$.

(b) The matrix δB of sensitivities of the hedging instruments to these scenarios is

$$\delta_i B_j = B_j\left(\mathcal{C}_i\right) - B_j\left(\mathcal{C}_0\right).$$

To avoid accidental colinearities between its rows or columns, one should always use more scenario than hedging instruments.

Finally, we translate the risk of the portfolio to the vector of hedge ratios with respect to the instruments in the hedging portfolio.

- The vector Δ of *hedge ratios* is calculated by minimizing

$$\mathcal{L}\left(\Delta\right) = \frac{1}{2}\,\|\delta B\,\Delta - \delta\Pi\|^2 + \frac{1}{2}\lambda\|Q\,\Delta\|^2.$$

Here, λ is an appropriately chosen small smoothness parameter (similar to the Tikhonov regularizer!), and Q is the smoothing operator (say, the identity matrix). Explicitly,

$$\Delta = \left((\delta B)^t\ \delta B + \lambda Q^t\,Q\right)^{-1}(\delta B)^t\ \delta\Pi,$$

where the superscript ^{t} denotes matrix transposition.

One can think of the component Δ_j as the sensitivity of the portfolio to the hedging instrument B_j. This method of calculating portfolio sensitivities is called the *ridge regression* method. It is very robust, and allows one to view the portfolio risk in a flexible way. One can use it together with both curve building techniques described above.

A B-splines and smoothing B-splines

We collect here a number of basic facts about B-splines. For a complete presentation, see [1].

A *spline* of degree d is a function $f(t)$ such that:

(a) $f(t)$ is piecewise polynomial of degree d. That means that one can partition the real line into non-overlapping intervals such that, on each of these intervals, $f(t)$ is a polynomial of degree d.

(b) $f(t)$ has $d-1$ continuous derivatives. That means that the polynomials mentioned above are glued together in a maximally smooth way.

Splines of low degree (such as $d = 3$, in which case they are called *cubic splines*) provide a convenient and robust framework for data interpolation.

A particular type of splines are *B-splines*. A B-spline of degree $d \geq 0$ is a function $f(t)$ of the form

$$f(t) = \sum_{k=-\infty}^{\infty} f_k B_k^{(d)}(t), \tag{23}$$

where $\left\{ B_k^{(d)}(t) \right\}$ is a family of basis functions defined as follows. We choose a sequence of *knot points*:

$$\ldots < t_{-1} < t_0 < t_1 < \ldots < t_k < \ldots, \tag{24}$$

and set

$$B_k^{(0)}(t) = \begin{cases} 1, & \text{if } t_k \leq t < t_{k+1}. \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

We then define recursively:

$$B_k^{(d)}(t) = \frac{t - t_k}{t_{k+d} - t_k} B_k^{(d-1)}(t) + \frac{t_{k+d+1} - t}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t). \tag{26}$$

Clearly, each $B_k^{(d)}(t)$ is a spline of degree d.

Here are some key properties of the basis functions:

$$B_k^{(d)}(t) \geq 0, \tag{27}$$

and

$$B_k^{(d)}(t) = 0, \tag{28}$$

if t lies outside of the interval $[t_k,\ t_{k+d+1}]$. Furthermore,

$$\sum_{k=-\infty}^{\infty} B_k^{(d)}(t) = 1. \tag{29}$$

One summarizes these three properties by saying that the basis functions $\left\{B_k^{(d)}(t)\right\}$ form a *partition of unity*.

Remarkably, differentiating and integrating of B-splines can be carried out in a recursive way as well. For the derivative we have the following recursion:

$$\frac{d}{dt} B_k^{(d)}(t) = \frac{d}{t_{k+d} - t_k} B_k^{(d-1)}(t) - \frac{d}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t). \tag{30}$$

The integral from $-\infty$ to t can be expressed in terms of a (finite!) sum as follows:

$$\int_{-\infty}^{t} B_k^{(d)}(\tau)\, d\tau = \sum_{i=k}^{\infty} \frac{t_{k+d+1} - t_k}{d+1} B_i^{(d+1)}(t), \tag{31}$$

and thus

$$\int_a^b B_k^{(d)}(\tau)\, d\tau = \int_{-\infty}^{b} B_k^{(d)}(\tau)\, d\tau - \int_{-\infty}^{a} B_k^{(d)}(\tau)\, d\tau. \tag{32}$$

Owing to these recursive properties, B-splines can be easily and robustly implemented in computer code.

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