1. (15 points) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

(a) Find A^{-1}

(b) Find the null space of A.

(c) Use part (a) to find a solution to

$$x + 2y + 3z = -1$$

$$y + 4z = 1$$

$$5x + 6y = -1$$

2. (20 points) Let $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(a) If $v = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, find $[v]_B$

(b) If $[v]_B = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, find v.

(c) If A is the matrix whose columns are the *first two only* vectors in B and let A be the matrix for a linear transformation T, find the $range(T)$.

3. (20 points) Recall that $\mathbb{P}_2$ is the space of all polynomials of degree 2 or lower. Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ by the formula

$$T(at^2 + bt + c) = a(2 - t)^2 + b(2 - t) + c.$$

(a) Find the matrix representation M of T with respect to the basis $\{1, t, t^2\}$.

(b) Show that $M^2 = I$. (*Hint: This can be done even without finding T in* (3a). *Consider* $T \circ T$.)

(c) Conclude that the only possible eigenvalues for M are ± 1. Find a linearly independent set of eigenvectors.

(d) For each $\mathbf{x}$ in (3c), find the corresponding $p(t) \in \mathbb{P}_2$ and evaluate $T(p(t))$ to see why these are called *eigenfunctions*.

4. (20 points) Let $\beta = \{1 + 2x, x - x^2, x + x^2\}$

(a) Show β is a basis for P_2.

(b) Let $T : P_2 \to P_2$ defined by $T(ax^2 + bx + c) = 2ax + b$ (that is the differential operator). Is T a linear transformation? Justify your answer. (Note, I have defined this transformation into P_2.

(c) Find the matrix of the linear transformation.

(d) Find $P_{(\alpha \leftarrow \beta)}$ where α is the standard basis for P_2. Recall that

$$[x]_\alpha = P_{(\alpha \leftarrow \beta)}[x]_\beta$$

5. (10 points) Pick one of the following to prove below:

(a) $Nul(A) = Nul(A^T A)$

(b) Let M_{22} be the vector space of all 2×2 matrices, and define $T : M_{22} \to M_{22}$ by $T(A) = A + A^T$. Prove that T is a linear transformation.

(c) Let A be an $n \times n$ - matrix and let $W = \{v \in R^n | Av = \lambda v\}$. Show that W is a subspace of R^n.

6. (20 points)

(a) Find an orthogonal basis for Col $\begin{bmatrix} 1 & 2 \\ -1 & 7 \\ 1 & 2 \end{bmatrix}$.

(*It need not be orthonormal.*)

(b) For $W = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix} \right\}$, find $W^{\perp}$

True or False

7. (2 points each) No partial credit given. No work need be shown.

   (a) ____Two vectors are linearly independent if one is not a scalar multiple of the other.

   (b) ____If A is diagonalizable, then there is a basis of eigenvectors of A.

   (c) ____If A does not have n distinct eigenvalues, then A is not diagonalizable

   (d) ____It is possible for the system $Ax = 0$ to have no real solution.

   (e) ____In the end, the only thing that matters is ...$Ax = b$.

Bonus: Tell me a joke. In order to receive any credit, it must make me grin.