1. (15 points) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

(a) Find $A^{-1}$

(b) Find the null space of $A$.

(c) Use part (a) to find a solution to

\begin{align*}
x + 2y + 3z &= -1 \\
y + 4z &= 1 \\
5x + 6y &= -1
\end{align*}
2. (20 points) Let \( B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \)

(a) If \( v = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \), find \([v]_B\)

(b) If \([v]_B = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\), find \(v\).
(c) If $A$ is the matrix whose columns are the *first two only* vectors in $B$ and let $A$ be the matrix for a linear transformation $T$, find the $\text{range}(T)$. 
3. (20 points) Recall that \( \mathbb{P}_2 \) is the space of all polynomials of degree 2 or lower. Define a linear transformation \( T : \mathbb{P}_2 \to \mathbb{P}_2 \) by the formula

\[
T(at^2 + bt + c) = a(2 - t)^2 + b(2 - t) + c.
\]

(a) Find the matrix representation \( M \) of \( T \) with respect to the basis \( \{1, t, t^2\} \).

(b) Show that \( M^2 = I \). (Hint: This can be done even without finding \( T \) in (3a). Consider \( T \circ T \).)
(c) Conclude that the only possible eigenvalues for $M$ are $\pm 1$. Find a linearly independent set of eigenvectors.

(d) For each $x$ in (3c), find the corresponding $p(t) \in P_2$ and evaluate $T(p(t))$ to see why these are called eigenfunctions.
4. (20 points) Let $\beta = \{1 + 2x, x - x^2, x + x^2\}$

(a) Show $\beta$ is a basis for $P_2$.

(b) Let $T : P_2 \to P_2$ defined by $T(ax^2 + bx + c) = 2ax + b$ (that is the differential operator). Is $T$ a linear transformation? Justify your answer. (Note, I have defined this transformation into $P_2$.)
(c) Find the matrix of the linear transformation.

(d) Find $P(\alpha \leftarrow \beta)$ where $\alpha$ is the standard basis for $P_2$. Recall that

$$[x]_{\alpha} = P(\alpha \leftarrow \beta)[x]_{\beta}$$
5. (10 points) Pick one of the following to prove below:

(a) \( \text{Nul}(A) = \text{Nul}(A^T A) \)
(b) Let \( M_{22} \) be the vector space of all \( 2 \times 2 \) matrices, and define \( T : M_{22} \to M_{22} \) by 
\[ T(A) = A + A^T. \]
Prove that \( T \) is a linear transformation.
(c) Let \( A \) be an \( n \times n \) - matrix and let \( W = \{ v \in \mathbb{R}^n | Av = \lambda v \} \). Show that \( W \) is a subspace of \( \mathbb{R}^n \).
6. (20 points)

(a) Find an orthogonal basis for $\text{Col} \begin{bmatrix} 1 & 2 \\ -1 & 7 \\ 1 & 2 \end{bmatrix}$.

(It need not be orthonormal.)
True or False

7. (2 points each) No partial credit given. No work need be shown.

(a) ____ Two vectors are linearly independent if one is not a scalar multiple of the other.

(b) ____ If \( A \) is diagonalizable, then there is a basis of eigenvectors of \( A \).

(c) ____ If \( A \) does not have \( n \) distinct eigenvalues, then \( A \) is not diagonalizable

(d) ____ It is possible for the system \( Ax = 0 \) to have no real solution.

(e) ____ In the end, the only thing that matters is \( ... Ax = b \).

Bonus: Tell me a joke. In order to receive any credit, it must make me grin.