(1) (Pre-calculus Review Set Problems 80 and 124.)

(a) Determine if each of the following statements is True or False. If it is true, explain why. If it is false, give a counterexample.

(i) If $a$ and $b$ are real numbers and $2a^5b^7 = 3ab$, then $2a^4b^6 = 3$. 

(ii) When solving $x^2(x - 2)^3 = (x - 2)^3$, we get $x^2 = 1$, so the solutions are $x_1 = 1, x_2 = 1$. 

(b) Simplify and write the following expression without negative exponents. Show your work.

\[
\frac{6^{-1}r^{-3}r^2}{r^5}
\]
(2) Find the derivatives of the given functions. You must show your work. But you do not have to simplify your answers.

(a) \( z(t) = \sqrt{t} (t^2 + t + 5) \)

(b) \( h(x) = \left( 5x^2 - \frac{x^2}{\sqrt{x} + 1} \right)^7 \)

(c) (6 points) Find \( \frac{dy}{dx} \) implicitly: \( e^{x^2}y = x + y \)
(3) (a) (9 points) The graph below is the derivative of a function $f(x)$, $f'(x)$. Answer each of the following questions. You do not need to explain.

(i) $f(x)$ is increasing on the interval(s) _________________________

(If you are not sure about the coordinates of the end points, an estimate will do.)

(ii) $f(x)$ is concave up on the interval(s) _________________________.

(iii) The function $f(x)$ achieves its absolute maximum value on the interval $[-4, 4]$ at $x = \underline{\text{ }}$.

(b) (6 points) Write a parameterization for the line segment from $(2, -5)$ and $(-1, 3)$.
(c) (6 points) Consider the function: \( f(x) = \begin{cases} 
|x - 1|, & 0 \leq x \leq 2 \\
3 - x, & 2 < x \leq 4 
\end{cases} \).

(i) Sketch the graph of \( f(x) \). Please label clearly the value of \( f \) at \( x = 0, 1, 2, 3, \) and 4.

(ii) Find the average value of \( f(x) \) on the interval \([0, 4]\).

(d) (6 points) Find the equation of the tangent line (in the form of \( y = mx + b \)) at the point \((2, 1)\) to the curve defined by the parametric equations

\[ x = 2t, \quad y = t^3. \]
Find each of the following limits when it exists, write DNE otherwise. Show your work.

(a) \( \lim_{x \to \infty} \frac{13 + 2x - 4x^4 + 3x^4}{2x - 5x^2 - 5x^4} \)

(b) \( \lim_{x \to a^+} \frac{x - a}{\sqrt{x^2 - a^2}} \), where \( a > 0. \)

(c) \( \lim_{x \to \infty} \frac{(\ln(x))^2}{x^2} \)
(5) Consider \( f(x) = xe^{-2x^2} \). Answer each of the following questions. You must show all work.

(a) Find \( f'(x) \) and all critical point(s) of \( f(x) \).

(b) Determine the interval(s) where \( f(x) \) is increasing and decreasing. State the \( x \)-coordinate(s) of the point(s) where \( f \) achieves its local maximum or/and local minimum.

(c) Find \( f''(x) \) and all inflection point(s) of \( f(x) \).
(6) The figure below shows the curve \( y = \sqrt{x} \), and a rectangle with its upper-left corner on the curve, its sides parallel to the axes, its left end at \( x = a \), and its right end at \( x = b \). Let \( b \) be fixed as \( b = 20 \). Find the value of \( a \) such that the rectangle has the maximum possible area. What is that maximum possible area? Show your work and give exact values.
Useful formulas

- The derivative of a function
  \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

- Some rules of differentiation
  \[ \frac{d}{dx} (cf(x)) = cf'(x) \]
  \[ \frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \]
  \[ \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \]
  \[ \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \]

- Differentiation formulas

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<thead>
<tr>
<th>( \frac{d}{dx} (x^n) = nx^{n-1} )</th>
<th>( \frac{d}{dx} (e^x) = e^x )</th>
<th>( \frac{d}{dx} (a^x) = (\ln a) a^x )</th>
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<tr>
<td>( \frac{d}{dx} (\ln x) = \frac{1}{x} )</td>
<td>( \frac{d}{dx} (\sin(x)) = \cos x )</td>
<td>( \frac{d}{dx} (\cos(x)) = -\sin x )</td>
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<td>( \frac{d}{dx} (\sec(x)) = \sec x \tan x )</td>
<td>( \frac{d}{dx} (\cot(x)) = -\csc^2 x )</td>
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<td>( \frac{d}{dx} (\arcsin(x)) = \frac{1}{\sqrt{1 - x^2}} )</td>
<td>( \frac{d}{dx} (\arccos(x)) = -\frac{1}{\sqrt{1 - x^2}} )</td>
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<td>( \frac{d}{dx} (\tanh(x)) = \frac{1}{\cosh^2(x)} )</td>
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- **The linear approximation** of a function \( f \) at \( a \) is given by
  \[ f(x) \approx f(a) + f'(a)(x - a) \]

- **Derivative of the inverse function** If \( f \) is a one-to-one differentiable function with inverse function \( f^{-1} \) and \( f'(f^{-1}(a)) \neq 0 \), then the inverse function of \( f \) is differentiable at \( a \) and
  \[ \frac{d}{dx} (f^{-1}(a)) = \frac{1}{f'(f^{-1}(a))} \]
• Parametric Equations for a straight line: An object moving along a line through the point \((x_0, y_0)\), with \(\frac{dx}{dt} = a\) and \(\frac{dy}{dt} = b\) has parametric equations

\[ x = x_0 + at, \quad y = y_0 + bt. \]

The slope of the line is \(m = \frac{b}{a}\).

• The instantaneous speed of a moving object is defined to be

\[ v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \]

The quantity \(v_x = \frac{dx}{dt}\) is the instantaneous velocity in the \(x\)-direction; \(v_y = \frac{dy}{dt}\) is the instantaneous velocity in the \(y\)-direction. The velocity vector \(\vec{v}\) is written \(\vec{v} = v_x \hat{i} + v_y \hat{j}\).

• For parametric curves,

\[
\text{Slope of curve} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{\frac{dy}{dx}}{\frac{dx}{dt}}\right).
\]

• Fundamental Theorem of Calculus: If \(f\) is continuous on the interval \([a, b]\) and \(f(t) = F'(t)\), then

\[ \int_a^b f(t) \, dt = F(b) - F(a). \]

• The average value of a function \(f\) on an interval \([a, b]\) is equal to \(\frac{1}{b - a} \int_a^b f(x) \, dx\).

• Comparison of Definite Integrals: If \(f\) is continuous and \(m \leq f(x) \leq M\) for \(a \leq x \leq b\), then \(m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)\).

• Basic integration formulas:

1. \(\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1)\)
2. \(\int \frac{1}{x} \, dx = \ln |x| + C\)
3. \(\int e^x \, dx = e^x + C\)
4. \(\int a^x \, dx = \frac{a^x}{\ln(a)} + C\)
5. \(\int \sin(x) \, dx = -\cos(x) + C\)
6. \(\int \cos(x) \, dx = \sin(x) + C\)
7. \(\int \sec^2(x) \, dx = \tan(x) + C\)
8. \(\int \csc^2(x) \, dx = -\cot(x) + C\)
9. \(\int \sec(x) \tan(x) \, dx = \sec(x) + C\)
10. \(\int \csc(x) \cot(x) \, dx = -\csc(x) + C\)
11. \(\int \frac{1}{x^2 + 1} \, dx = \tan^{-1}(x) + C\)
12. \(\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1}(x) + C\)