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G63.2020/G22.2421: Numerical Methods II | Handout #6 | Mar. 30, 2000

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Problem Set 8

In the following implementations, the three functions $a, b, f \in C[0,1]$ are defined by the formulae

$$a(x) = 1 + x\ln(2+x), \tag{1}$$
$$b(x) = 1/(2+x), \tag{2}$$
$$f(x) = (1+x^2)(1+\cos(15x)) \tag{3}$$
$$\gamma = 1,\ \beta = 2, \tag{4}$$

Problem 1

Follow the procedures described on page 136–139 to construct finite element solution of the boundary value problem

$$-(a(x)u'(x))' + b(x)u(x) = f(x),\quad x \in (0,1), \tag{5}$$
$$u(0) = \alpha, \tag{6}$$
$$u(1) = \beta \tag{7}$$

on the (non-equispaced) grid $x_0 = 0$, $x_j = (j-1/2)h,\ j = 1,2,\ldots,m$, $x_{m+1} = 1$; $h = 1/m$. For the implementation, we further require
(1) Set $\phi_0(x) = \alpha + (\beta - \alpha)x$ so that the inhomogeneous boundary conditions (6), (7) are satisfied;
(2) Rewrite (8.3) as

$$u_m(x) = \phi_0(x) + v(x), \tag{8}$$

where $v(x)$ is a piecewise linear function of the form

$$v(x) = \sum_{l=1}^{m} v(x_l)\phi_l(x). \tag{9}$$

Thus, $\gamma_j = v(x_j)$ now is the value of v at the interior grid points, whereas $\phi_j(x)$ is the finite element basis function that is piecewise linear and is equal to 1 at $x = x_j$ and zero at the rest of the grip points (it is known as the hat function). Obviously, the function v satisfies the homogeneous boundary conditions $v(0) = v(1) = 0$.
(3) Break up integrals over $[0,1]$ to the form

$$\int_{x_{j-1}}^{x_j} g(x)dx,\quad \int_{x_j}^{x_{j_1}} g(x)dx \tag{10}$$

and evaluate in each subinterval by trapezoidal rule so that the relative error in the approximation is $O(h^2)$.

(a) Form the matrix and the righ hand side of the linear system for Galerkin method;
(b) Given $\alpha = 2,\ \beta = 3$, numerically solve the linear equations for $v(x_j),\ j = 1,2,\ldots,m$.
(c) Show rate of convergence of $u_m(x)$ at $x = 0.5173601$ for $m = 20, 40, 80, 160$.
(d) Plot the numerical solution $u_m(x)$ in $[0,1]$ for $m = 80$.

Problem 2

Modify $\phi_0,\ \phi_0,\ \phi_m$ and use Galerkin method to solve the problem

$$\begin{aligned} -(a(x)u'(x))' + b(x)u(x) &= f(x), \quad x \in (0,1), & (11)\\ u'(0) &= 0, & (12)\\ u'(1) + \gamma u(1) &= \beta, & (13) \end{aligned}$$

ON THE NON-EQUISPACED GRID $x_0 = 0$, $x_j = (j-1/2)h,\ j = 1, 2, \ldots, m$, $x_{m+1} = 1$; $h = 1/m$. For implementation we require
(1) Choose ϕ_0 as a suitable constant function to satisfy the inhomogeneous boundary condition (13);
(2) Modify ϕ_1, the first hat function, only in the subinterval $[x_0, x_1]$ such that it is still a linear function there, and that it satisfies the homogeneous boundary condition (12);
(3) Modify ϕ_m, the last hat function, only in the subinterval $[x_m, x_{m+1}]$ such that it is still a linear function there, and that it satisfies the homogeneous boundary condition $u'(1) + \gamma u(1) = 0$;

Remark: After the modifications of ϕ_1 and ϕ_m, they should still be continuous functions in $[0,1]$.

Remark: There is no need to modify ϕ_j for $j = 2, 3, \ldots, m-1$ since all these basis functions satisfy the two homogeneous boundary conditions at $x = 0$ and $x = 1$ automatically.

(a) Form the matrix and the righ hand side of the linear equations for Galerkin method. Note that these equations are for the interior points only, thus there are m equations;
(b) For parameters specified in (1)–(4), numerically solve the linear system for $v(x_j),\ j = 1, 2, \ldots, m$ for $m = 160$. Plot the numerical solution $u_m(x)$ in $[0,1]$.

Problem 3

Make necessary changes and additions to the procedures described on page 143–144 to show that a smooth solution u of the variational problem

$$\begin{aligned} J(u) &= \min_{v\in H} J(v), \ u \in H, & (14)\\ J(v) &= \int_0^1 \left\{ \frac{1}{2}a(x)[v'(x)]^2 + \frac{1}{2}b(x)[v(x)]^2 - f(x)v(x) \right\} dx + a(1)v(1)\left[\frac{1}{2}\gamma v(1) - \beta\right] & (15) \end{aligned}$$

is a solution of the boundary value problem (11)–(13). In the above, it is assumed that a, b, f are continuous functions, and that a, b, γ are positive. Furthermore, H and H° are chosen as the same space defined by

$$H = H^\circ = \left\{ v \ \middle| \ \int_0^1 v^2(x)dx, \int_0^1 [v'(x)]^2 dx < \infty \ \right\}, \qquad (16)$$

so that there is no boundary conditions imposed on H or H°; hence the term "natural boundary condition".

Problem 4

Follow the procedures described on page 145 to construct finite element solution, based on the Ritz method, of the variational problem (14), (15) (equivalently, of the boundary value problem (11)–(13)), on the equispaced points $x_j = (j-1)h$, $j = 1, 2, \ldots, m$, $h = 1/(m-1)$ (please take note of the spacing). Note that the resulting linear equations are on both the interior and the two end-points $x = 0,\ x = 1$. Also note that ϕ_0 is zero or absent for this approach.
(a) Form the matrix and the righ hand side of the linear equations for Ritz method;
(b) For parameters specified in (1)–(4), numerically solve the linear system for $u_m(x_j) = v(x_j),\ j = 1, 2, \ldots, m$ for $m = 160$. Plot the numerical solution $u_m(x)$ in $[0,1]$.