

Problem Set 1

Use double precision in numerical computation

Problem 1

- (a) Apply the method of proof of Theorems 1.1 and 1.2 to establish convergence of the backward (or implicit) Euler's method.
(b) Do the same thing for the implicit midpoint rule (1.12).

Problem 2

Consider the scalar Riccati equation

$$\begin{cases} y' = i \cdot k \{y^2 - [1 + \mu \cdot \sin(5t)]\}, & 0 \leq t \leq 2\pi, \\ y(0) = 1, \end{cases} \quad (1)$$

where k and μ are constants, $i = (0, 1)$ is the imaginary unit. For each of the two sets of parameters (i) $k = 3, \mu = 0.5$ (ii) $k = 30, \mu = 0.94$, implement the following three schemes on a equispaced grid with $h = 2\pi/N$, $t_n = n \cdot h$, and check their rates of convergence numerically.

- (a) Forward (or explicit) Euler's method;
(b) Trapezoidal rule (solve the nonlinear equation with Newton's method);
(c) The scheme given by the formula

$$y_{n+1} = y_n + i \cdot hk \{y_n \cdot y_{n+1} - [1 + \mu \cdot \sin(5t_{n+1/2})]\} \quad (2)$$

- (d) Is the problem stiff in the case $k = 30, \mu = 0.94$?

Problem 3

Let A be a nonsingular matrix and $\{X_k\}$, $k = 0, 1, \dots$, be a sequence of matrices satisfying

$$X_{k+1} := X_k + X_k(I - AX_k) \quad (3)$$

(Schulz's method).

- a) Show that $\|I - AX_0\| < 1$ is sufficient to ensure convergence of $\{X_k\}$ to A^{-1} (where $\|\cdot\|$ denote an induced matrix norm). Furthermore, $E_k = I - AX_k$ satisfies

$$E_{k+1} = E_k E_k. \quad (4)$$

- b) Show that Schulz's method is locally quadratically convergent (in the same induced norm).

- c) If, in addition $AX_0 = X_0A$, then for all $k > 0$

$$AX_k = X_kA. \quad (5)$$

- d) Derive the Newton's method for the problem $F(X) = 0$ with $F(X) = I - AX$ and A nonsingular. Explain why it is not practical. Modify it to a practical one, and compare your result with Schulz's method.

Problem 4

Suppose that a polynomial of degree 200 has 100 real and simple roots in the interval $[-10, 10]$, separated one from another by a distance of at least 4×10^{-3} . Suppose further that you are to find the real roots to double precision. Describe a simple and effective procedure to solve the problem (no actual programming is required).