Matrix norms induced by vector norms on $\mathbb{R}^n$:

$$\|A\| = \max_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \frac{\|Av\|}{\|v\|} = \max_{\|v\|=1} \|Av\|$$

Pf. Let $\|A\|_* = \max_{\|v\|=1} \|Av\|$

(1) $\|A\| = \max_{v \neq 0} \frac{\|Av\|}{\|v\|} \geq \max_{\|v\|=1} \frac{\|Av\|}{\|v\|} = \max_{\|v\|=1} \|Av\|$

$= \|A\|_*$

(2) For $v \neq 0$

$$\frac{\|Av\|}{\|v\|} = \left\| A \frac{v}{\|v\|} \right\| \leq \max_{\|w\|=1} \|Aw\| = \|A\|_*$$

$$\Rightarrow \max_{v \neq 0} \frac{\|Av\|}{\|v\|} \leq \|A\|_*$$

Note
$\|I\| = 1$ in any induced norm.

Thm $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$, i.e. max absolute row sum.

Pf. $v \in \mathbb{R}^n$, $v \neq 0$

(1) $|(Av)_i| = \left| \sum_{j=1}^n a_{ij} v_j \right| \leq \sum_{j=1}^n |a_{ij}| |v_j|$ Δ ineq.

$\leq \|v\|_\infty \sum_{j=1}^n |a_{ij}|$

take max or i.

$$\Rightarrow \frac{\|Av\|_\infty}{\|v\|_\infty} \leq \cancel{\sum_{j=1}^n |a_{ij}|} \max_i \sum_{j=1}^n |a_{ij}| \neq \sum_{j=1}^n$$

$\Rightarrow \|A\|_\infty \leq \max_i \sum_{j=1}^n |a_{ij}|$

(2) ~~Let $v_j = sgn$~~

Note: $\exists\, m$ s.t. $\max_i \sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{mj}|$

Let $\tilde{v}_j = sgn(a_{mj})$. Then $\|v\|_\infty = 1$

$$\|A\tilde{v}\|_\infty = \max_i \left|\sum_{j=1}^n a_{ij}\tilde{v}_j\right| \geq \left|\sum_{j=1}^n a_{mj}\tilde{v}_j\right|$$

$$= \sum_{j=1}^n |a_{mj}|$$

$$= \max_i \sum_{j=1}^n |a_{ij}|$$

$\Rightarrow$ ~~max~~

$$\|A\|_\infty = \max_{\|v\|=1} \|Av\|_\infty \geq \|A\tilde{v}\|_\infty \geq \max_i \sum_j |a_{ij}|$$

$$\Rightarrow \|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$$

<u>Thm</u> $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$ max absolute column sum

[<u>Cor</u>: $\|A\|_\infty = \|A^T\|_1$]

<u>Thm</u> $\|A\|_2 = \max_{1\leq i\leq n} \sqrt{\lambda_i}$; $\lambda_i \geq 0$ are e.values of A^TA.

Condition numbers

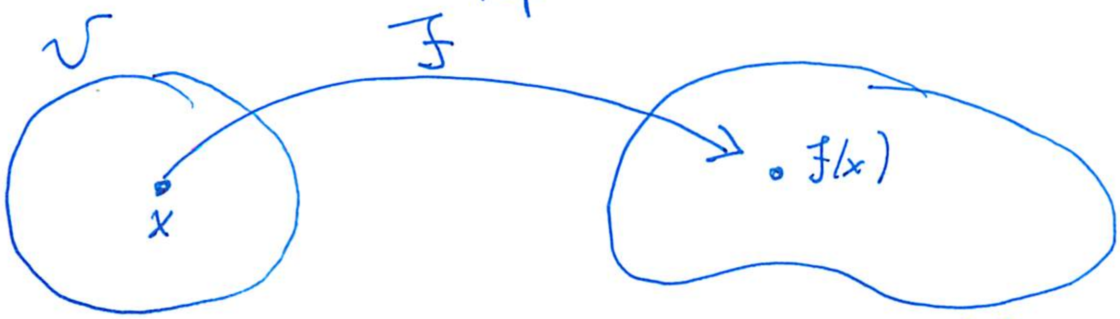
Sensitivity of output to input.

Consider 2 normed linear spaces [like $\mathbb{R}^n$, $\mathbb{R}^m$]

V & W

Let $f : (V; \|\cdot\|_V) \to (W, \|\cdot\|_W)$

↑ input ↑ output



Absolute local condition # of f (at x)

$$C^a_x[f] = \sup_{\substack{\delta x \in V \\ \delta x \neq 0 \\ \delta x \to 0 \\ x+\delta x \in V}} \frac{\|f(x+\delta x) - f(x)\|_W}{\|\delta x\|_V}$$

over vanishing $\|\cdot\|_V$ balls about x.

Relative local cond. # [$x \neq 0$, $f(x) \neq 0$]

$$C^r_x[f] = \sup_{\substack{\delta x \in V \\ \delta x \neq 0 \\ \to 0 \\ x+\delta x \in V}} \frac{\|f(x+\delta x) - f(x)\|_W / \|f(x)\|_W}{\|\delta x\|_V / \|x\|_V}$$

$C_x[f] \gg 1$ ill-conditioned

$C_x[f] \sim 1$ well-conditioned.

Condition # of a matrix inverse $A^{-1} \in \mathbb{R}^{n\times n}$

[$y = f(x)$ above $\iff$ $x = A^{-1}b$]

Choose a ~~nor~~ vector norm on $\mathbb{R}^n$, & and hence an induced matrix norm on $\mathbb{R}^{n\times n}$.

$$C_b^r\left[A^{-1}\right] = \sup_{\substack{\delta b \in \mathbb{R}^n \\ \delta b \neq 0 \\ \delta b \to 0}} \frac{\|A^{-1}(b+\delta b) - A^{-1}b\| / \|A^{-1}b\|}{\|\delta b\| / \|b\|}$$

$$= \sup_{\delta b \neq 0} \left[\frac{\|A^{-1}\delta b\| / \|A^{-1}b\|}{\|\delta b\| / \|b\|} = \frac{\|A^{-1}\delta b\|}{\|\delta b\|} \cdot \frac{\|b\|}{\|A^{-1}b\|} \right]$$

$\vdots$

$\|A^{-1}\delta b\| \leq \|A^{-1}\| \cdot \|\delta b\|$ since $\|A^{-1}\|$ is an induced norm & hence compatible.

Further $b = A(A^{-1}b) \Rightarrow \|b\| \leq \|A\| \cdot \|A^{-1}b\|$

or $\dfrac{\|b\|}{\|A^{-1}b\|} \leq \|A\|$

$$\Rightarrow C_b^r\left[A^{-1}\right] \leq \|A\| \cdot \|A^{-1}\| = \kappa(A^{-1})$$

~~$\kappa(A)$ depends upon~~

Note that for a n.s. matrix A

$\kappa(A) = \kappa(A^{-1}) =$ condition # of a n.s. matrix.

Note: $\kappa(A)$ is independent ~~of~~ of b, and hence is global (i.e. for any b)

Note: $\kappa(A)$ depends upon the norm chosen

Note: $1 = \|I\| = \|AA^{-1}\| \le \|A\| \cdot \|A^{-1}\| = \kappa(A)$

Theorem $A \in \mathbb{R}^{n \times n}$ n.s., $b \in \mathbb{R}^n$

Let $Ax = b$ & $A(x + \delta x) = b + \delta b$

Then $\dfrac{\|\delta x\|}{\|x\|} \le \kappa(A) \dfrac{\|\delta b\|}{\|b\|}$

Pf. Previous inequality.

$$\delta x \rightarrow \frac{\|A^{-1}\delta b\|}{\|\delta b\|} \cdot \frac{\|b\|}{\cancel{\|\delta b\|}\ \|A^{-1}b\|} \le \kappa(A^{-1}) = \kappa(A)$$

$\|A^{-1}b\| \leftarrow x$

That is, the condition number gives the amplification factor of ~~error~~ relative error in the ~~in~~ input ~~to the~~ $\|\delta b\|/\|b\|$ to the ~~output~~ ~~$\|\delta x\|$~~ relative error in the output $\|\delta x\|/\|x\|$.

Example

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} ; \quad K_2(A) \approx 524$$

Solve $Ax = b$, $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $\|x\|_2 = 48$

Let $\delta b = \begin{pmatrix} 0 \\ \varepsilon \\ 0 \end{pmatrix}$

$$\frac{\|\delta x\|_2}{\|x\|_2} = \frac{\|\delta x\|_2}{48} \leq 524 \cdot \frac{\varepsilon}{1}$$

If $\varepsilon = 10^{-4}$, then we have very little accuracy # $\frac{\|\delta x\|_2}{\|x\|} \leq \frac{1}{2}$