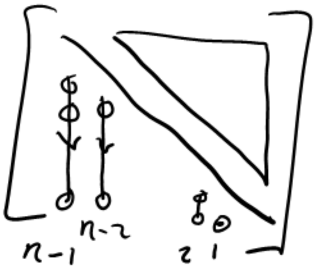
So, more generally, from Gaussian elimination, we have for $A \in \mathbb{R}^{n\times n}$

$$L_N L_{N-1} \cdots L_1 A = U,$$



$$L_j = I + \mu_{rs} \underbrace{e^r (e^s)^T}_{\text{rank-one matrix}}$$

$1 \le s < r \le n$

$N = 1 + 2 + \cdots + (n-1)$
$= n(n-1)/2$

$$L_j^{-1} = I - \mu_{rs} e^r (e^s)^T$$

$\Rightarrow A = L_1^{-1} \cdots L_N^{-1} U$ $\quad$ $\det L_i = 1 \Rightarrow L_i$ n.s.

$= LU$ $\quad$ L_i^{-1} unit lower Δ

w. L a unit lower Δ matrix

Solving $Ax = b$? If $A = LU$ & A is n.s.

$$L\underbrace{Ux}_{=y} = b$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ \vdots & & \ddots & \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

① solve $Ly = b$

by recursively by forward substitution

$$y_1 = b_1$$
$$y_2 = b_2 - l_{21} y_1 = b_2 - l_{21} b_1$$
$$\vdots$$

② Given y, solve

$Ux = y$ by backward substitution.

matlab $\quad$ `y = L\b; x = U\y;` or `x = U\(L\b);`

This is what matlab does in `x = A\b`.

Direct computation of L & U:

$$\begin{bmatrix} 1 & & & 0 \\ l_{21} & 1 & & \\ \vdots & & 1 & \\ l_{n1} & \cdots & & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{21} & \cdots \\ & \ddots & \vdots \\ & & u_{nn} \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}$$

$$a_{ij} = \sum_{k=1}^{n} l_{ik} u_{kj} = \sum_{k=1}^{\min\{i,j\}} l_{ik} u_{kj} \ ; \text{ or}$$

$$a_{ij} = \sum_{k=1}^{j} l_{ik} u_{kj} \qquad 1 \le j < i \le n \quad \text{below the diagonal}$$

$$= \sum_{k=1}^{i} l_{ik} u_{kj} \qquad 1 \le i \le j \le n \quad \text{at \& above the diagonal}$$

or

$$a_{ij} = \sum_{k=1}^{j-1} l_{ik} u_{kj} + l_{ij} u_{jj} \qquad 1 \le j < i \le n$$

$$= \sum_{k=1}^{i-1} l_{ik} u_{kj} + \underbrace{l_{ii}}_{=1} u_{ij} \qquad 1 \le i \le j \le n$$

or

$$l_{ij} = \frac{1}{u_{jj}} \left[a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right] \qquad i = 2(1)n, \ j = 1(1)(i-1)$$

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \qquad i = 1(1)n, \ j = i(1)n$$

$\boxed{i=1}$ $u_{1j} = a_{1j}$, $j = 1(1)n$ $\quad$ 1st row of U

$\boxed{j=1}$ $l_{11} = 1$, $l_{i1} = \dfrac{a_{i1}}{u_{11}}$, $i = 2(1)n$, 1st column of L

$\boxed{i=2}$ $u_{2j} = a_{2j} - l_{21} u_{1j}$ $\quad j = 2(1)n$

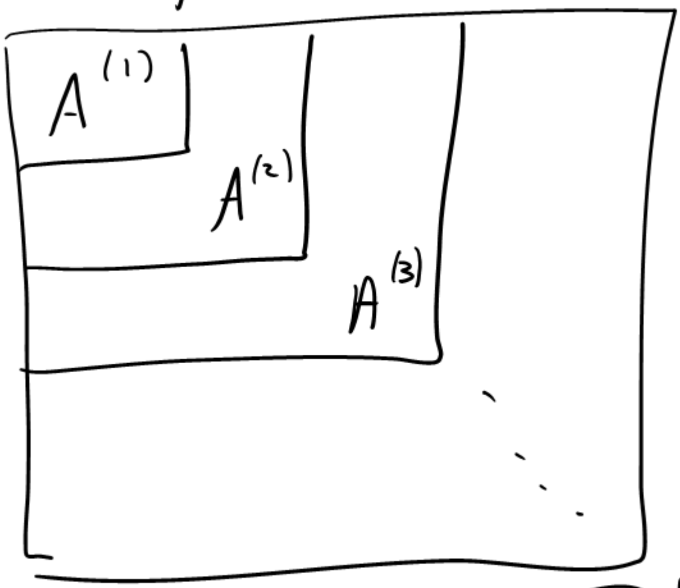
$\boxed{j=2}$ $l_{i2} = \dfrac{1}{u_{22}} (a_{i2} - l_{i1} u_{12})$ $\quad i = 3(1)n$ …

So, we can determine L & U recursively it seems, but requires $u_{jj} \neq 0$, $j = 1(1)(n-1)$

When does it all work?

$n \geq 2$

<u>Defn</u> Given $A \in \mathbb{R}^{n \times n}$, its "leading principle submatrix" of order k, $A^{(k)} \in \mathbb{R}^{k \times k}$ is $a^{(k)}_{ij} = a_{ij}$ for $1 \leq i \leq k < n$, $1 \leq j \leq k < n$



$A^{(k)}$ coincides w A on the upper left.

<u>Thm</u> Given $A \in \mathbb{R}^{n \times n}$, $n \geq 2$.

If <u>every</u> $A^{(k)}$, $k = 1(1)(n-1)$ is n.s. then A can be "factorized" as

$$A = LU$$

w. L unit lower Δ & U upper Δ.

<u>Pf.</u> By induction on $n \geq 2$

<u>$n=2$</u> $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ By hypothesis

$A^{(1)} = a \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix}\begin{pmatrix} u & v \\ 0 & \eta \end{pmatrix} = \begin{pmatrix} u & v \\ u\ell & \ell v + \eta \end{pmatrix}$$

$\Rightarrow u = a$, $v = b$, $u\ell = c \Rightarrow \ell = c/a$

$d = \ell v + \eta \Rightarrow \eta = d - cb/a$ ✓

<u>$n = k+1$</u> | Assume $A \in \mathbb{R}^{(k+1)\times(k+1)}$ has nonsingular leading principal matrices of order k & below, & so $A^{(k)} = L^{(k)}U^{(k)}$

$$A = \left[\begin{array}{c|c} A^{(k)} = L^{(k)}U^{(k)} & b \\ \hline - c^T - & d \end{array}\right] = \left[\begin{array}{c|c} L^{(k)} & 0 \\ \hline - m^T - & 1 \end{array}\right]\left[\begin{array}{c|c} U^{(k)} & v \\ \hline - 0 - & \eta \end{array}\right]$$

$\Rightarrow$ ① $A^{(k)} = L^{(k)}U^{(k)}$ ✓ ② $L^{(k)} v = b$

$\det(L^{(k)}) = 1 \Rightarrow (L^{(k)})^{-1}$ exists. $v = (L^{(k)})^{-1} b$

③ $m^T U^{(k)} = c^T$

$\Rightarrow (U^{(k)})^T m = c$

$0 \neq \det(A^{(k)}) = \det(L^{(k)})\det(U^{(k)})$

$= \underline{1} \cdot \det((U^{(k)})^T) \Rightarrow (U^{(k)})^T$ is n.s.

& $m = ((U^{(k)})^T)^{-1} c$ ✓

④ $m \cdot v + \eta = d$ ✓

<u>Nice.</u>

Do we ever have to worry about this.

<u>Yes.</u>, trivially

$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & 5 & 4 \end{pmatrix}$ does not have an LU decomposition

$\tilde{A} = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 1 & 1 \\ -1 & 5 & 4 \end{pmatrix}$ does.

Pivoting – Building row exchanges into matrix decomposition.

Permutation matrices

$P \in \mathbb{R}^{n \times n}$ are matrices w. one and only one one per row & column.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad 2 \text{ in } \mathbb{R}^{2\times 2}$$

$3 \times 2 \times 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dots \qquad 3! \text{ in } \mathbb{R}^{3\times 3}$$

etc.

Properties (1) $P_3 = P_1 P_2$ is again a permutation matrix

(2) $\det(P) = \pm 1$ (why?)

(3) Are product of interchange matrices

(4) Inverses are permutation matrices.