Now, let's assume $f'(x_*) = \alpha \neq 0$

(1) Then, since $f'(x)$ is assumed continuous in a nghd of x_*, $\exists\ \delta_1 > 0$ s.t.

$$|f'(x)| > \frac{1}{2}\alpha \quad \forall\ |x - x_*| < \delta_1$$

(2) Since $f''(x)$ is also continuous,

$\exists\ \delta_2 > 0$, $M > 0$, s.t.

$$|f''(x)| \le M \quad \forall\ |x - x_*| < \delta_2$$

Let $\delta_3 = \min(\delta_1, \delta_2)$. Then

$$\left| \frac{f''(x)}{2f'(y)} \right| \le C = \frac{M}{\alpha} (> 0)$$

$\forall\ x, y$ s.t. $|x - x_*| \le \delta_3$ & $|y - x_*| \le \delta_3$

Here is an outline of a proof.

How do we pick x_0?

1 Pick $x_0 \ni |x_0 - x_*| \le \delta_3$

Then $|x_* - x_0| \le \frac{1}{C}$

$\Rightarrow |x_* - x_1|$

Now, let's choose x_0 ... $C|x_* - x_0| \le \frac{1}{2}|x_* - x_0|$

Now, let's assume $|F'(x_*)| = \alpha \neq 0$.

(1) Since F' is continuous, $\exists\ \delta_1 > 0$ s.t.

$$|f'(x)| \geq \frac{\alpha}{2} \quad \forall\ |x - x_*| \leq \delta_1$$

(2) Since $F''(x)$ is continuous,

$\exists\ \delta_2 > 0$, $M > 0$ s.t.

$$|F''(x)| \leq M \quad \forall\ |x - x_*| \leq \delta_2$$

Let $\delta_3 = \min(\delta_1, \delta_2)$. Then $\exists\ 0 < \bar{C} < \infty$

s.t. $C(y,z) = \left| F''(y) / 2f'(z) \right| < \bar{C}$

$\forall\ |y - x_*| \leq \delta_3$ & $|z - x_*| \leq \delta_3$

For N.M. we have

$$C_k = |F''(\eta_k) / 2f'(x_k)| = C(\eta_k, x_k) \text{ w.}$$

$$|\eta_k - x_*| \leq |x_k - x_*|$$

Outline of convergence proof

It all comes down, as it must, to the choice of x_0.

$$|x_1 - x_*| = C(\eta_0, x_0)|x_0 - x_*|^2$$

(1) Choose $|x_0 - x_*| \le \delta_3$

$$\Rightarrow C(\eta_0, x_0) \le \overline{C}$$

(2) Let $\delta = \min(\delta_3, 1/2\,\overline{C}) > 0$

$$\Rightarrow \overline{C}|x_0 - x_*|^2 \le \tfrac{1}{2}|x_0 - x_*|$$

$$\Rightarrow |x_1 - x_*| = C_0|x_0 - x_*|^2 \le \tfrac{1}{2}|x_0 - x_*| \le \tfrac{1}{2}\delta$$

At the next step we have

$$|x_2 - x_*| = C(\eta_1, x_1)|x_1 - x_*|^2$$

w. $|\eta_1 - x_*| \le |x_1 - x_*| \le \frac{1}{2}\delta < \delta$

$$\Rightarrow C_1 = C(\eta_1, x_1) \le \overline{C}$$

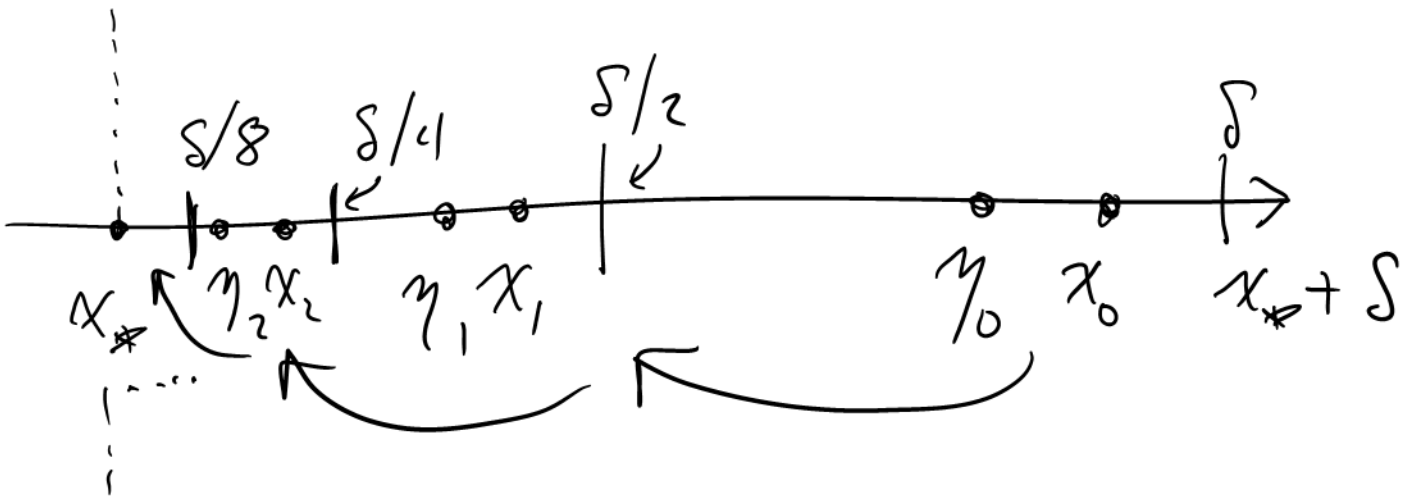
$$\Rightarrow |x_2 - x_*| \le \tfrac{1}{2}|x_1 - x_*| \le \left(\tfrac{1}{2}\right)^2 \delta < \delta$$ by the same argument

and so on. At each iteration, η_k & x_k stay within the δ-nghd of x_*, & C_k can be bded by $\overline{C}$.

Iterating this we have

$$|\eta_k - x_*| \leq |x_k - x_*| \leq \left(\tfrac{1}{2}\right)^k \delta$$

Hence, $\eta_k, x_k \longrightarrow x_*$ as $k \to \infty$.

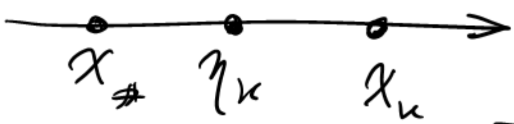


<u>Defn</u> A seq. $\{x_k\}$, w. $x_k \to x_*$, is said to converge to x_* w. <u>at least</u> <u>order</u> q iff $\exists$ $\{\varepsilon_k\}$; $\varepsilon_k \to 0$, and $\mu > 0$ s.t.

$$|x_k - x_*| \le \varepsilon_k \quad k = 0, 1, \ldots \quad \& \quad \lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^q} \to \mu$$

$$|x_{k+1} - x_*| = C_k |x_k - x_*|^2$$

$$\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|^2} = \lim_{k\to\infty} \left| \frac{f''(\eta_k)}{2f'(x_k)} \right|$$



$x_k \to x_*$

$\Rightarrow \eta_k \to x_*$

$$= \left| \frac{f''(x_*)}{2f'(x_*)} \right| = \mu \quad \text{i.e. } q = 2.$$

<u>Note</u> If $f''(x_*) = 0$, then the defn above apparently breaks down. This is silly since $\mu = 0$ reflects a super-quadratic rate of conv. for N.M. when $f''(x_*) = 0$ & the defn says "at least order q" (i.e. quadratic here.)

If $f''(x_*) = 0$, $q = 3$ (cubic)

<u>Q</u>: Will N.M. converge if $f'(x_*) = 0$?

Let's work a simple example.

Let $f(x) = x^p$; $p \geq 1$ & integer

$$f'(x) = p\,x^{p-1} \qquad [\,x_* = 0\,]$$

<u>N.M.</u> $$x_{k+1} = x_k - \frac{x_k^p}{p\,x_k^{p-1}} = \left(1 - \frac{1}{p}\right) x_k$$

i.e. $$|x_{k+1} - x_*| = \left(1 - \frac{1}{p}\right) |x_k - x_*|$$

This is only linear converge for $p > 1$ & for $p = 1$, converge to exact solution in 1 iteration for any x_0

<u>Q</u>: Why not go to a higher order approx. to f @ $x = x_k$?

<u>N.M.</u> $$M(x; x_k) = f(x_k) + f'(x_k)(x - x_k)$$

x_{k+1} satisfies $M(x_{k+1}; x_k) = 0$

<u>NNM</u>: $$Q(x; x_k) = f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2} f''(x_k)(x - x_k)^2$$

w. $Q(x_{k+1}; x_k) = 0$

(i) Must solve quadratic eqn w. multiple roots; which is x_{n+1}?

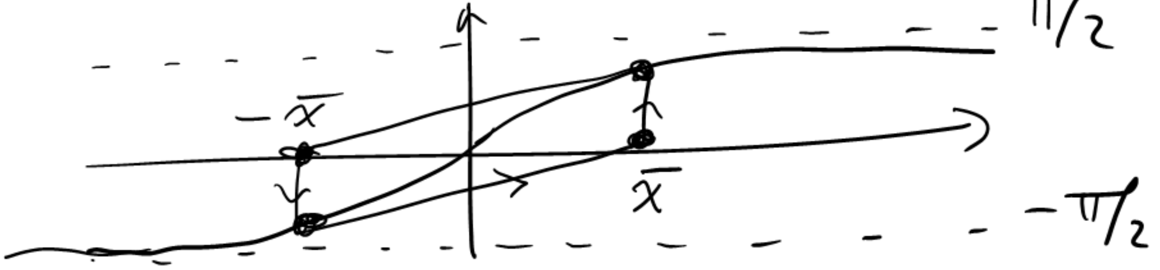
(ii) Req's higher deriv. info.

(iii.) Do not generalize well to higher dimensions

(iv) N.M. is pretty fast and almost certainly more robust.

<u>A weakness of N.M.</u> — x_0 must be suff'ly close for convergence.

Consider $f(x) = \arctan x$



$$f'(x) = \frac{1}{1+x^2} \; ; \; x_{k+1} = x_k - (1+x_k^2)\,\mathrm{atan}\, x_k$$

Find $\bar{x}$ s.t. $-\bar{x} = \bar{x} - (1+\bar{x}^2)\,\mathrm{atan}\,\bar{x}$

$$h(\bar{x}) = 2\bar{x} - (1+\bar{x}^2)\,\mathrm{atan}\,\bar{x}$$

$$h'(\bar{x}) = 2 - 2\bar{x}\,\mathrm{atan}\,\bar{x} - 1$$
$$= 1 - 2\bar{x}\,\mathrm{atan}\,\bar{x}$$

Use N.M $\bar{x} = 1.3917\ldots$

$$x_0 > \bar{x} \Rightarrow |x_k| \to \infty$$

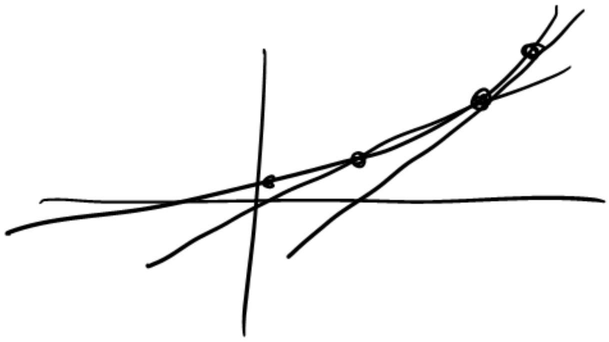
$$x_0 < \bar{x} \Rightarrow |x_k| \to 0$$

What if derivatives are hard to get?

Secant method

$$f'(x_k) \simeq \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$x_{k+1} = x_k - f(x_k)\,\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$



Need x_0 & x_1

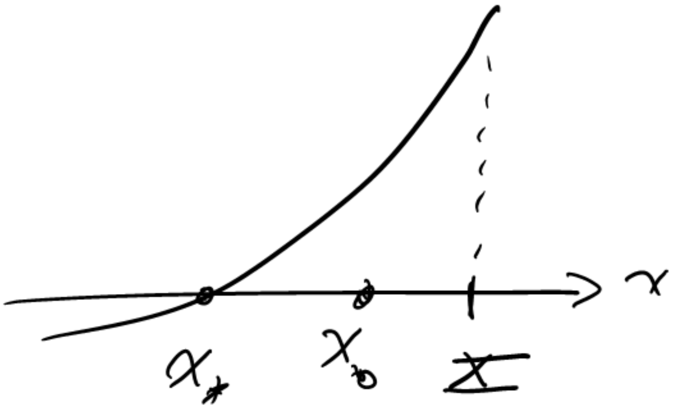
Thm Use conditions very similar to N.M., S.M converges but slightly weaker. Instead of $q=2$,

$$q = \frac{1}{2}(1+\sqrt{5}) \simeq 1.6. \quad \text{super-linear}$$

S.M. generalizes in a very interesting way to systems of eqns through methods called quasi-Newton or Broyden schemes.

Read about bisection. As I implied, shows linear convergence w. $\mu = 1/2$.

In some cases, the "close enough" condition on x_0 can be relaxed



On $[x_*, \bar{x}]$

$f', f'' > 0$

conv. for any

$x_0 \in [x_*, \bar{x}]$

What if we have a system of eqns?

Find $\underline{\vec{x}}_* = (x_{1*}, x_{2*}, \dots, x_{n*}) \in \mathbb{R}^n$ s.t.

$$f_1(x_{1*}, x_{2*}, \dots, x_{n*}) = 0$$
$$\vdots$$
$$f_n(x_{1*}, x_{2*}, \dots, x_{n*}) = 0$$

$\vec{F} = (f_1, \dots, f_n)$

$\vec{F}(\underline{x}_*) = 0$

$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$

Approx. $\vec{F}(\vec{x})$ at $\underline{x} = \underline{x}_k$ by

$$\vec{M}(\underline{x}; \underline{x}_k) = \vec{F}(\underline{x}_k) + \underline{\underline{J}}(\underline{x}_k)(\underline{x} - \underline{x}_k)$$

N.M. in $\mathbb{R}^n$: $\vec{M}(\underline{x}_{k+1}; \underline{x}_k) = 0$

$$\Longleftrightarrow \boxed{\underline{\underline{J}}(\underline{x}_k)(\underline{x}_{k+1} - \underline{x}_k) = -\vec{F}(\underline{x}_k)}$$

$\underline{\underline{J}}$ is a $\mathbb{R}^{n\times n}$ matrix $J_{ij} = \partial f_i / \partial x_j$